

Consider the following argument:

Premises: All human beings are rational

Some animals are human beings

Conclusion: Hence, some animals are rational.

If we let $H(x)$ mean " x is a human being", $R(x)$ mean " x is rational" and $A(x)$ mean " x is an animal", then we can rewrite this argument as

$$\frac{(\forall x) (H(x) \Rightarrow R(x)) \quad (\exists x) (A(x) \wedge H(x))}{\therefore (\exists x) (A(x) \wedge R(x))}$$

" $\forall x$ " means "for all x "

" $\exists x$ " means "there exists at least one x such that"

Qn: Why is this argument valid?

This type of argument is known as a syllogism and was studied by Aristotle. This argument cannot be justified by Propositional Logic.

Consider now, another argument:

No student is a reptile

No reptile is rational

\therefore No student is rational

Qn: Is this argument valid?

No.

The Syntax of the Predicate Logics:

The alphabet of a PDL consists of

1. Individual variables:

$x, y, z, x_1, x_2, x_3 \dots$

2. A set of constant letters:

nonempty $a, b, c, a_1, a_2, a_3, \dots$

3. A set of Predicate letters (or Relation symbols)

$P, Q, R, A_k^{(n)}$ $n, k = 1, 2, 3, \dots$

$n = \text{arity}$ of the Predicate letter

4. A set of Function letters (or operation symbols)

$f, g, h, f_k^{(n)}$ $n, k = 1, 2, 3, \dots$

$n = \text{arity}$ of the function letter

5. Primitive connectives: \neg, \Rightarrow

6. Primitive Quantifier: \forall

7. Parentheses and comma: $(,), ", ;$

The other connectives and quantifier will be introduced as follows:

$A \wedge B$ is used for $\neg (\neg A \Rightarrow \neg B)$

$A \vee B$ " " " $(\neg A \Rightarrow B)$

$A \Leftrightarrow B$ " " " $(A \Rightarrow B) \wedge (B \Rightarrow A)$

$(\exists x) A$ is used for $\neg ((\forall x) (\neg A))$

The terms of the Pred. Logics are defined recursively as follows:

1. Any individual variable or individual constant is a term
2. If f is a function letter with arity n and t_1, \dots, t_n are terms then $f(t_1, \dots, t_n)$ is a term

Roughly speaking the terms of Pred. Logics are like the "nouns" of a natural language. To get sentences we need to add "verbs".

The formulas of the Pred. Logics are defined recursively as follows:

1. If R is a Predicate letter of arity n and t_1, \dots, t_n are terms then $R(t_1, \dots, t_n)$ is a formula

[Such formulas are called atomic formulas]

2. If A and B are formulas and x is a variable, then

$(\forall A)$, $(A \Rightarrow B)$, and $((\forall x)A)$ are also formulas.

In the form $((\forall x) A)$, A is called the scope of the quantifier " $(\forall x)$ ".

Note: A does not have to contain the variable x in order for us to form the formula $((\forall x)A)$

Examples : To specify a Predicate Logic, all we need to do is to specific the constant letters, Predicate letters, and function letters because every Predicate Logic will contain all the individual variables and the connectives, quantifiers, parentheses and the comma.

Note : We need to have at least one predicate letter, otherwise we won't be able to form any formulas.

Let $\{a, b\}$, $\{P, Q, R\}$, $\{f\}$ be the constant, Predicate and function letters of a PDL, Λ with P being unary, Q & R binary and f binary
Then the following are terms of Λ

a, b, x

$f(a, b), f(b, a), f(x, x)$

$f(f(a, b), f(b, a))$

The following are formulas of Λ

$P(a), P(x)$

$R(a, b), R(x, y)$

$(\forall x) R(a, b)$

$(\forall x) P(x)$

$(\forall x) (P(x) \Rightarrow Q(x, x))$

$((\forall x) P(x)) \Rightarrow \neg((\forall y) R(y, y))$

Bound and Free variables

Let A be a formula and x be a variable.

The x can occur in several places in A . An occurrence of x is said to be bound in A if the " x " is next a quantifier as in " $\forall x$ " or " $\exists x$ ". or if the " x " is within the scope of a " $\forall x$ " or " $\exists x$ " as in " $((\forall x) P(x, y))$ " or $((\exists x) \neg R(x))$

In all other cases the " x " is said to be free in A

Examples

$$1. P(x, x) \Rightarrow (\forall y) R(y)$$

↑ ↑ ↑ ↑
free free bound bound

$$2. (\forall x) (P(x, y) \Rightarrow R(x))$$

↑ ↑ ↙ ↑
bound bound free bound

$$3. ((\exists x) P(x, y)) \Rightarrow (\forall z) Q(x, z)$$

↑ ↑ ↙ ↑ ↑ ↑
bound bound free bound free free

We usually write $A(x_1, \dots, x_k)$ instead of just A to emphasize that x_1, \dots, x_k may occur free in A . Then we can write $A(t_1, \dots, t_n)$ to mean the formula obtained by replacing all the free occurrences of x_1, \dots, x_k (if any) by t_1, \dots, t_k respectively.

Ex. Let $A(x, y)$ be $(\forall x)(P(x, y) \Rightarrow R(x))$ and t_1 be "a" and t_2 be "f(b)". Then
 $A(t_1, t_2)$ is $(\forall x)(P(x, f(b)) \Rightarrow R(x))$

Terms which are free for a variable :

Lect #7 Consider the formula $P(x)$

Here the "x" is free. So we can substitute anything for it and we won't really change the formula. So we can say that the term "y" is free for x in $P(x)$.

But now look at the formula $(\forall y) P(x)$

Here "x" is still free. But if we substitute "y" for "x" we will drastically change the formula to $(\forall y) P(y)$. So "y" is not free for x in $(\forall y) P(x)$. But lots of terms such as "a", $f(a)$ or "z" are free for "x" in $(\forall y) P(x)$.

Def. let A be a formula and t be a term.

We say that t is free for x in A if no free occurrence of x lies within the scope of a " $\forall y$ " quantifier or " $\exists y$ " quantifier with y being a variable in t.

So the term $g(y, z)$ is not free for x in $(\forall y) P(x)$ because when we substitute $g(y, z)$ for x — the "y" will be bound by the " $\forall y$ ".

Def. A structure U consists of a set D called its domain together with some relations on D , some functions on D and some distinguished elements of D

$$U = \langle D; \{R_i^U\}; \{f_i^U\}; \{a_i^U\} \rangle$$

The semantics of the Predicate Logics:

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Consider a Predicate Logic $\Lambda = \langle \{a_i\}, \{R_j\}, \{f_k\} \rangle$

Let A be a formula of Λ

Qn: What does " A " mean?

Roughly speaking the formulas of Λ are like algebraic equations which involves unknowns such as x, y, z , etc. If we think of x, y, z , etc., as numbers — these equations may be true or false depending on what values we assign to x, y, z , etc.

Example " $x^2 + y^2 = 2xy$ " is true if $x=1 \& y=1$
 " $x^2 + y^2 = 2xy$ " is false if $x=1 \& y=2$

There are some equations which are true no matter what values we assign to the unknowns. Such equations are called identities.

Example " $(x+y)^2 = x^2 + y^2 + 2xy$ " is true
 for any assignment of numbers to $x \& y$.
So we usually write $(x+y)^2 = x^2 + y^2 + 2xy$.

By assigning values from a specified structure to the "unknowns" in A , we can determine whether or not the formula A will be true for that assignment. If the formula is true for all possible assignments from a specified structure we will say it is true in that structure.

Def. An interpretation of a Pred. Logic
 $\mathcal{N} = \langle \{R_i\}; \{f_j\}; \{a_k\} \rangle$ consists of a domain
 D and an assignment function φ .

To each n -ary Relation symbol R_i , φ assigns
 an n -ary relation $(R_i)^M \subseteq D^n$

To each n -ary function symbol f_j , φ
 assigns an n -ary function $(f_j)^M: D^n \rightarrow D$

To each constant symbol a_k , φ assigns
 an element $(a_k)^M \in D$.

An interpretation

$$M = \langle D, \{R_i^M\}, \{f_j^M\}, \{a_k^M\} \rangle$$

is actually just a structure which is
 compatible with \mathcal{N} .

Example :

Let $\mathcal{N} = \{A; f; a, b\}$ where A
 is a binary relation symbol & f is a
 binary function symbol.

Let $M = \langle N; \leq; +; 0, 1 \rangle$ $N = \{0, 1, 2, \dots\}$
 Then M , is an interpretation of \mathcal{N}

Consider the following formulas :

1. $A(a, b)$

2. $(\forall y) A(y, f(y, b))$

3. $(\exists x)(\forall y) A(x, y)$

4. $(\forall y)(\forall x) A(x, y)$

5. $A(x, y) \Rightarrow A(f(x, x), f(y, y))$

1. $0 \leq 1$? Yes. So #1 is true in M_1
2. $(\forall y)(y \leq y+1)$? Yes So #2 is true in M_1
3. $(\exists x)(\forall y)(x \leq y)$? Yes. Take $x=0$. So #3 is true in M_1 .
4. $(\exists y)(\forall x)(x \leq y)$? No. So #4 is false in M_1 .
5. $(x \leq y) \Rightarrow (x+x \leq y+y)$? True for all assignment of values to x & y .

Let $M_2 = \langle \mathbb{Z}; \leq; +; -1, 1 \rangle$. Then M_2 is another interpretation of Λ .

#1 $-1 \leq 1$? true in M_2

#2 $(\forall y)(y \leq y, 1)$? true in M_2

#3 $(\exists x)(\forall y)(x \leq y)$? false in M_2

#4. $(\exists y)(\forall x)(x \leq y)$? false in M_2

#5 $(x \leq y) \Rightarrow (x \cdot x \leq y \cdot y)$? True for some assignments
False for some assignments

We want to define what it means for a formula to be true in an interpretation.

Let A be a formula of $\Lambda = (\{R_i\}, \{f_j\}, \{g_k\})$ (31)
 and M be an interpretation of Λ .
 Let $D = \text{domain of } M$ and
 $\text{SEQ} = \text{set of infinite sequences of elements of } D$.

A typical element of SEQ will look like

$$\underline{s} = \langle s_1, s_2, s_3, \dots \rangle$$

First, given $\underline{s} \in \text{SEQ}$ we define the function s^* (that will assign an element of D to each term t) as follows:

1. If t is the variable x_j , let $s^*(t) = s_j$
2. If t is the constant a , let $s^*(t) = (a)^M$
3. If t is $f(t_1, \dots, t_n)$, then let
 $s^*(t) = f^M(s^*(t_1), \dots, s^*(t_n))$

We now define what it means for \underline{s} to satisfy the formula A as follows:

1. If A is $R(t_1, \dots, t_n)$ then \underline{s} satisfies A iff $R^M(s^*(t_1), \dots, s^*(t_n))$ holds
2. If A is $\neg B$, \underline{s} satisfies A iff \underline{s} does not satisfy B .
3. If A is $B \Rightarrow C$, then \underline{s} satisfies A iff \underline{s} does not satisfy B or if \underline{s} satisfies C .
4. If A is $(\forall x_i) B$, then \underline{s} satisfies A iff every sequence \underline{s}' that agrees with \underline{s} except for the i -th component perhaps, satisfies B .

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Intuitively speaking, a seq. $\underline{s} = \langle s_1, s_2, s_3, \dots \rangle$ satisfies A iff when we replace the free occurrences of each x_i by s_i , the resulting formula is true in M .

Def. We say that A is true in the interpretation M (and write $\models_M A$) if every seq. $\underline{s} \in \text{SEQ}$ satisfies A . We say A is totally false in M iff no $\underline{s} \in \text{SEQ}$ satisfies A .

Example: Let A be the formula

$$A(x_1, x_2) \Rightarrow A(f(x_1, x_1), f(x_2, x_2)).$$

Then A is true in the interpretation $M = \langle \mathbb{N}, \leq, +, 0, 1 \rangle$ because for any seq. $\underline{s} \in \text{SEQ}$ we can check that \underline{s} satisfies A .

Def. Let Γ be a set of formulas of a Pred. Logic \mathcal{L} . We say that an interpretation M of \mathcal{L} is a model of Γ if every formula in Γ is true in M .

Ex- Let $\Gamma = \{ A(a, b), (\forall y)(A(y, f(y, b))), (\exists x)(\forall y) A(x, y) \}$
 $= \{ \#1, \#2, \#3 \}$

Then $M = \langle \mathbb{N}, \leq, +, 0, 1 \rangle$ is a model of Γ .

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Some Properties of "true in M" (\vdash_M):

- I A is true in M iff $\neg A$ is totally false in M
- II It is not possible to have both $\vdash_M A$ and $\vdash_M \neg A$.
- III If $\vdash_M A$ and $\vdash_M A \Rightarrow B$, then $\vdash_M B$
 Suppose $\vdash_M A$ and $\vdash_M A \Rightarrow B$. Then
 \underline{s} satisfies A and $A \Rightarrow B$ for each $\underline{s} \in \text{SEQ}$
 So \underline{s} satisfies B for each $\underline{s} \in \text{SEQ}$
 Thus $\vdash_M B$.]
- IV $A \Rightarrow B$ is totally false in M iff $\vdash_M A$ and $\vdash_M \neg B$.
- V
 - a) \underline{s} satisfies $A \wedge B$ iff \underline{s} satisfies A & \underline{s} satisfies B
 - b) \underline{s} " $A \vee B$ iff " or "
 - c) \underline{s} " $A \Leftrightarrow B$ iff \underline{s} satisfies $A \wedge B$, or \underline{s} satisfies $\neg A \wedge \neg B$
 - d) \underline{s} " $(\exists x_i)A$ iff there is a seq. \underline{s}' (that differs from \underline{s} in at most the i -th component) which satisfies A .
- VI If A is a formula with x_1, \dots, x_n as its free variables then we define the closure of A by $c1(A) = (\forall x_1) \dots (\forall x_n) A$.
 - (a) $\vdash_M A$ iff $\vdash_M (\forall x_i) A$ (even if x_i is not free in A)
 - (b) $\vdash_M A$ iff $\vdash_M c1(A)$

VII If \mathcal{A} is an instance of a tautology, then $\vDash_M \mathcal{A}$ for any interpretation M .

Ex. $(A \wedge B) \Rightarrow A$ is a tautology.

$$\text{So } \vDash_M [P(x_1, x_2) \wedge Q(x_2)] \Rightarrow P(x_1, x_2).$$

VIII If the free variables of \mathcal{A} occur in the list x_1, \dots, x_n and two sequences s and s' agree on the 1-st, 2nd, ..., n -th component, then

s satisfies \mathcal{A} iff s' satisfies \mathcal{A} .

IX A formula is said to be closed if it has no free variables. If \mathcal{A} is a closed formula then $\vDash_M \mathcal{A}$ or $\vDash_M \neg \mathcal{A}$.

X If t is free for x_i in the formula $\mathcal{A}(x_i)$ then

$$\vDash_M ((\forall x_i) \mathcal{A}(x_i)) \Rightarrow \mathcal{A}(t) \quad \text{for any interpretation } M.$$

XI If x_i is not free in \mathcal{A} , then

$$\vDash_M ((\forall x_i) (\mathcal{A} \Rightarrow \mathcal{B})) \Rightarrow (\mathcal{A} \Rightarrow ((\forall x_i) \mathcal{B}))$$

for any interpretation M .

Logically valid formulas

Def. Let $\Lambda = \langle \{R_i\}; \{f_j\}; \{a_k\} \rangle$ be a Predicate Logic. A formula A of Λ is said to be logically valid if A is true in every interpretation M of Λ .

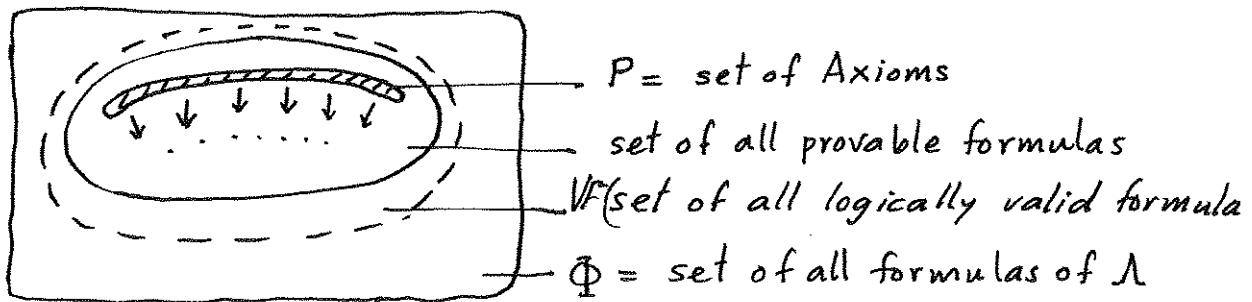
A is said to be satisfiable if there is an interpretation M such that A is satisfied by at least one sequence $\underline{s} \in \text{SEQ}$.

A is said to be a contradictory if A is totally false in every interpretation M of Λ .

Fact: 1. A is logically valid iff $\neg A$ is not satisfiable
 2. A is logically valid iff $\neg A$ is contradictory
 3. A is contradictory iff A is not satisfiable.

Qu: Given a formula A of a Pred. Logic Λ , how can we find out if A is logically valid?

Aus: Find an axiomatization of Λ and then use the method of "proofs" to find out. (Formal Ded. System)



Qu: Can we get all of VF from P? Yes!

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Lect. #9 A formal deductive system for Predicate Logics : K_0

$$K_0 = \langle \Sigma, \Phi, P, R \rangle$$

We already know what the alphabet Σ and what the set of formulas Φ are. The set of axioms P and the set of rules of inferences R are given below:

Axioms : P

$$(A1) A \Rightarrow (B \Rightarrow A)$$

$$(A2) (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$$

$$(A3) (\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$$

$$(A4) ((\forall x_i) A(x_i)) \Rightarrow A(t) \quad \text{provided } t \text{ is free for } x_i$$

$$(A5) ((\forall x_i)(A \Rightarrow B)) \Rightarrow (A \Rightarrow (\forall x_i) B) \quad \text{provided } x_i \text{ is not free in } A.$$

Rules of inference : R

(MP) Modus Ponens : From $A \& A \Rightarrow B$ infer B

(GEN) Generalization : From A infer $(\forall x_i) A$.

Explanations for $(A4) \& (A5)$

(A4) We need t to be free for x_i in $A(x_i)$

Let's take $A(x_1)$ to be $(\exists x_2)(x_1 \neq x_2)$

and t to be x_2 . If we apply (A4) we will

get $(\forall x_1)((\exists x_2)(x_1 \neq x_2)) \Rightarrow (\exists x_2)(x_2 \neq x_2)$ which

is not a valid formula. So t has to be free for x_i

(A5) Here is an example which makes (A5) seems reasonable

$$(\forall x)(a < b \Rightarrow x+a < x+b) \Rightarrow ((a < b) \Rightarrow (\forall x)(x+a < x+b))$$

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Proposition 1: Every formula of $\Lambda = \{\{R_i\}, \{f_j\}, \{g_k\}\}$ that is an instance of a tautology is provable in the F.D.S. K_0 .

Proof: let A be a formula which is an instance of a tautology. Then we can find a tautology T which co with constituent statement letters A_1, \dots, A_n such that A is obtained from T by replacing A_1, \dots, A_n with formulas A_1, \dots, A_n of Λ .

Now by the Completeness Theorem for Prop. Logic, we know that there is a proof of T in the F.D.S. L (because T is a tautology). If we replace the statement letters A_1, \dots, A_n by the formulas A_1, \dots, A_n in each line of this proof we will get a proof of A in K_0 . Notice that in the proof we will only use (A1)-(A3) & MP. We don't need (A4) & (A5) to prove A .

Recall that a F.D.S. S is said to be consistent if there is no formula A such that both A and $\neg A$ are provable in S .

Proposition 2: The F.D.S. K_0 is consistent.

Proof: For each formula A , let $h(A)$ be the expression obtained by erasing all quantifiers and terms (together with the associated commas & parentheses) from A .

Ex. If A is $(\neg(\forall x) B(x_4, a, x_1)) \Rightarrow C(x_4, a)$
Then $h(A)$ is $\neg B \Rightarrow C$.

Then $h(A)$ is essentially a statement form with the relation symbols playing the role of statement letters. Also it is clear that

$$h[\neg A] \text{ is } \neg h[A], \text{ and}$$

$$h[A \Rightarrow B] \text{ is } h[A] \Rightarrow h[B].$$

So each of the axioms (A1)–(A5) will be transformed by h into a tautology.

Moreover if $h[A]$ and $h[A \Rightarrow B]$ are tautologies we know that $h[B]$ will be a tautology. So MP transforms tautologies into tautologies. And finally $h[(\forall x_i) A]$ is just $h[A]$. So GN will also transform tautologies into tautologies. Hence if A is provable in K_0 , then $h[A]$ will be a tautology.

Now suppose both A & $\neg A$ were provable in K_0 . Then $h[A]$ and $\neg h[A]$ will both be tautologies. But this is clearly impossible. Hence there is no A such that both A & $\neg A$ are provable in K_0 . Hence K_0 is consistent.

Remarks: In the proof of Proposition 2, the transformation λ amounts to interpreting M in a domain with exactly one element. If A is provable in K_0 , then by the Soundness theorem which we will prove next, A will be true in this interpretation. Prop. 2 then says that there is no formula A such that both A & $\neg A$ are true in this interpretation. This we know from Property II of "true in M "

Theorem 3: If A is provable in the F.D.S. K_0 , then A is logically valid. (Soundness theorem)

Proof: Since $(A1) - (A3)$ are all instances of tautologies, it follows by Property III of "true in M " that $(A1) - (A3)$ are true in all interpretations M . So $(A1) - (A3)$ are all logically valid. Also $(A4)$ & $(A5)$ are logically valid in the same way because of Properties X and XI. Finally from Properties III & IV(a), we see that the rules of inference MP & GEN preserve logical validity (i.e. we can only get logical validities from logical validities by using these two rules.)

Now if A is provable in K_0 , then we can get A from the Axioms by using the two rules of inference. So if A is provable in K_0 , then A must be logically valid.

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Lec 1 #10 In the Prop. Logic the Deduction theorem says:
If $\Gamma, A \vdash_L B$, then $\Gamma \vdash_L (A \Rightarrow B)$.

In the Pred. Logics we would expect the Deduction theorem to say:

If $\Gamma, A \vdash_{K_0} B$, then $\Gamma \vdash_{K_0} (A \Rightarrow B)$.

Unfortunately, because of the new rule of inference GEN, this result is not true.

Example Let $A = A(x_1)$ and $B = (\forall x_1) A(x_1)$.

Then $A \vdash_{K_0} B$. Here is the proof:

1. $A(x_1)$ Hypothesis

2. $(\forall x_1) A(x_1)$ by GEN rule from 1.

But we do not have $\vdash_{K_0} A \Rightarrow B$. Indeed, if we did, then by the soundness theorem $A \Rightarrow B$ will be logically valid. But

$$A(x_1) \Rightarrow (\forall x_1) A(x_1) \quad (*)$$

is clearly not logically valid. Just take the interpretation $M = \langle N, R \rangle$ where R is the unary relation which says that an element is prime. The formula $(*)$ is not satisfied by the sequence $\langle 2, 5_2, 5_3, \dots \rangle$. So $A \Rightarrow B$ is not true in M . Hence $A \Rightarrow B$ could not be logically valid.

So, what's wrong here? The problem is with the GEN. rule. If we include it we can't get a nice Deduction Theorem & if we exclude it we handicap ourself when doing proofs.

Let Γ be a set of formulas and B_1, \dots, B_n be a deduction from Γ . Also let A be a formula in Γ . We say that B_i depends on A in this deduction if

1. B_i is A and the justification for this line of the deduction is that $B_i \in \Gamma$, or
2. B_i is justified by using MP or GEN on some previous B_j 's, one of which depends on A .

Ex. Consider the deduction

$$A, ((\forall x_1) A) \Rightarrow b \vdash (\forall x_1) b$$

| | | | |
|----|-------|-----------------------------------|------------------|
| 1. | B_1 | A | Hypothesis |
| 2. | B_2 | $(\forall x_1) A$ | from 1. by GEN |
| 3. | B_3 | $((\forall x_1) A) \Rightarrow b$ | Hypothesis |
| 4. | B_4 | b | from 1 & 2 by MP |
| 5. | B_5 | $(\forall x_1) b$ | from 4 by GEN |

In this deduction B_1, B_2, B_4 & B_5 all depend on A . But B_3 does not depend on A (even though the formula B_3 contains A as a part).

Proposition 4: If $\Gamma \vdash_k B$ and B does not depend on A in some deduction of B from Γ & A , then $\Gamma \vdash_{K_0} B$.

Proof: See textbook p. 74, Prop. 2.4

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Theorem 5 (Deduction theorem for the Pred. Logics)

Suppose $\Gamma, A \vdash_{K_0} B$ and in some deduction of B from $\Gamma \& A$, we never applied the GEN rule to any formula $b(x_i)$ (which depends on A and with x_i being a free variable of A) to get $(\forall x_i) b(x_i)$. Then $\Gamma \vdash_{K_0} (A \Rightarrow B)$

Proof: Let $B_1, \dots, B_n = B$ be a deduction of B from $\Gamma \& A$ in which the conditions above are satisfied. We have to show that $\Gamma \vdash (A \Rightarrow B)$. We will show by induction that $\Gamma \vdash (A \Rightarrow B_i)$ for each $i=1, \dots, n$.

Case (i) : B_i is an axiom or $B_i \in \Gamma$

In this case we get a proof of $A \Rightarrow B_i$ from Γ as follows:

1. $B_i \Rightarrow (A \Rightarrow B_i)$. Axiom (A1)
2. B_i Axiom or Hyp.
3. $A \Rightarrow B_i$ from 1. & 2. by MP

Case (ii) : B_i is A

In this case $A \Rightarrow B_i$ is just $A \Rightarrow A$ which is an instance of a tautology. So by Prop. 1 there is a proof of $A \Rightarrow B_i$ from \emptyset . So we, of course, get $\Gamma \vdash A \Rightarrow B_i$

Case (iii) : B_i is obtained from B_j and $B_k = (B_j \Rightarrow B_i)$ for some $j, k < i$, by MP.

In this case we know that

$$\Gamma \vdash (A \Rightarrow B_j) \quad \& \quad \Gamma \vdash A \Rightarrow (B_j \Rightarrow B_i)$$

by the ind. hypothesis. But by axiom (A2)

$$\emptyset \vdash (A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$$

So by MP

$$\Gamma \vdash (A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)$$

and by MP again

$$\Gamma \vdash A \Rightarrow B_i.$$

Case (iv) : B_i is obtained from B_j by GEN
for some $j < i$, $B_i = (\forall x_k) B_j$
and x_k is not free in B_j .

In this case we get from axiom A5

$$\emptyset \vdash (\forall x_k)(A \Rightarrow B_j) \Rightarrow (A \Rightarrow (\forall x_k) B_j)$$

and $\Gamma \vdash A \Rightarrow B_j$ by the induction hy.

So $\Gamma \vdash (\forall x_k)(A \Rightarrow B_j)$ by GEN.

$\therefore \Gamma \vdash A \Rightarrow (\forall x_k) B_j$ by MP.

i.e. $\Gamma \vdash A \Rightarrow B_i$

Case (v) : B_i is obtained from B_j by GEN for
some $j < i$, $B_i = (\forall x_k) B_j$, and
 B_j does not depend on A

In this case we know that

$$\Gamma \vdash A \Rightarrow B_j \quad \text{by the induction hyp.}$$

$$\therefore \Gamma \vdash B_j \quad \text{by Proposition 4}$$

$$\therefore \Gamma \vdash (\forall x_k) B_j \quad \text{by applying GEN.}$$

$$\therefore \Gamma \vdash B_i$$

Now by axiom (A1) $\emptyset \vdash B_i \Rightarrow (A \Rightarrow B_i)$

So by MP, $\Gamma \vdash A \Rightarrow B_i$.

This completes the proof.

Theories in the Predicate logic :

Let $\Lambda = \langle \{R_i\}, \{f_j\}, \{g_k\} \rangle$ be the language of a Predicate Logic. We already defined what is the set Φ of all formulas of Λ .

A theory \mathcal{T} in this Predicate Logic (or a First-Order Theory) is just a set of formulas of Λ , (i.e., \mathcal{T} is any subset of Φ).

Def. A first order theory \mathcal{T} is said to be consistent if it has a model (i.e., if we can find an interpretation M such that every formula of \mathcal{T} is true in M).

Def. A first order theory \mathcal{T} is said to be recursively axiomatizable (finitely axiomatizable) if we can find a ^{consistent} F.D.S. S with a recursive (resp. finite) set of axioms such that if $A \in \mathcal{T}$ then A is provable in S .

Def. A first order theory \mathcal{T} is said to be decidable if there is an algorithm which can tell us whether or not a given formula A is in \mathcal{T} .

Can get natural theories in 3 ways :

1. $VF(\Lambda) =$ set of all logically valid formulas of Λ .
2. $PF_S(\Lambda) =$ set of all formulas of Λ that are provable in S .
3. $TF_M(\Lambda) =$ set of all formulas of Λ that are true in M .

Lec. # 11A Some derived rules of inference:

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1. PARTICULARIZATION RULE: (Rule A4)

If t is a term which is free for x in $A(x)$,
then $(\forall x) A(x) \vdash A(t)$

2. EXISTENTIAL RULE (Rule E4)

If t is a term that is free for x in $A(x, t)$
and $A(t, t)$ arise from $A(x, t)$ by replacing all
free occurrences of x by t , then
 $A(t, t) \vdash (\exists x) A(x, t)$.

3. NEGATION RULES: $\neg\neg A \vdash A$

(Elimination)

$$A \vdash \neg\neg A$$

(Introduction)

4. CONJUNCTION RULES:

$$A \wedge B \vdash A, \quad \neg(A \wedge B) \vdash (\neg A) \vee (\neg B)$$

Introd.

$$A, B \vdash A \wedge B$$

Elim.

5. DISJUNCTION RULES:

$$A \vee B, \neg A \vdash B, \quad \neg(A \vee B) \vdash (\neg A) \wedge (\neg B)$$

Elim.

$$A \vdash A \vee B, \quad B \vdash A \vee B$$

Introd.

6. CONDITIONAL RULES $\neg(A \Rightarrow B) \vdash A; \quad A \Rightarrow B, \neg B \vdash \neg A$ Elim.

Elim.

Introd.

7. BICONDITIONAL RULES: $A \Leftrightarrow B, A \vdash B; \quad A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$

8. PROOF BY CONTRADICTION: If a proof show $\Gamma, \neg A \vdash \text{Contradiction}$
involves no application of GEN on a free variable of A ,
then $\Gamma \vdash A$

Rule A4 : $(\forall x)A(x) \vdash A(t)$ provided t is free for x in $A(x)$

Proof:

1. $(\forall x)A(x)$ Hyp.
2. $(\forall x)A(x) \Rightarrow A(t)$ Axiom A4
3. $A(t)$ from 1. & 2. by MP

Rule AE4 : $A(t,t) \vdash (\exists x)B(x,t)$ provided t is free for x in $A(x,t)$

Proof:

1. $B(t,t)$ Hyp.
2. $(\forall x)(\neg B(z,t)) \rightarrow \neg B(t,t)$ Ax. 4
3. $B(t,t) \rightarrow \neg(\forall x)(\neg B(z,t))$ inst. of taut. from 2
 $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
4. $\neg(\forall x)(\neg B(z,t))$ from 1. & 3. by MP

But 4. is just an abbreviation of $(\exists x)B(x,t)$
so we are done

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(C is for choice)

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Rule C : It is common in Mathematics to reason in the following way. Suppose we have proved a formula of the form $(\exists x)B(x)$. Then we say, let b be an object such that $B(b)$ holds. We then continue the proof and arrive at a formula which does not involve b .

Example : Suppose we wish to prove
 $(\exists x)(B(x) \Rightarrow L(x))$, $(\forall x)B(x) \vdash (\exists x)L(x)$

1. $(\exists x)(B(x) \Rightarrow L(x))$ Hyp.
2. $(\forall x)B(x)$ Hyp.
3. $B(b) \Rightarrow L(b)$ for some b from 1 by Rule C
4. $B(b)$ from 2 by Rule A4
5. $L(b)$ from 3 & 4 by MP
6. $(\exists x)L(x)$ from 5 by rule E4

In general any formula that can be proved using a finite number of arbitrary choices can also be proved without any such acts of choices.

The Rule that permits us from going to from $(\exists x)B(x)$ to $B(b)$ is called Rule C.

(see p. 81-82 of the textbook.) We have the following fact

Fact: If $\Gamma \vdash_{C,K} B$, then $\Gamma \vdash_K B$

Here $\vdash_{C,K}$ means that you can use all the axioms of K & rules of inf. in K as well as the rule of inf. c.

Examples of First Order F.D.S.

Def. A First Order F.D.S. is any F.D.S. obtained by adding more axioms to the F.D.S. K_0 . The new axioms are called non-logical axioms.

Examples:

1. Partial Order F.D.S.: $\Lambda = \{ < \}$

$$K_0 + (a) (\forall x) (\neg(x < x))$$

$$(b) (\forall x)(\forall y)(\forall z) (x < y \wedge y < z \Rightarrow x < z)$$

2. Group Theory F.D.S.: Λ is $\{\approx, +, 0\}$

$$K_0 + (a) (\forall x)(\forall y)(\forall z) (x + (y + z) \approx (x + y) + z)$$

$$(b) (\forall x) (0 + x \approx x)$$

$$(c) (\forall x) (\exists y) (y + x \approx 0)$$

$$(d) (\forall x) (x \approx x)$$

$$(e) (\forall x)(\forall y) (x \approx y \Rightarrow y \approx x)$$

$$(f) (\forall x)(\forall y)(\forall z) (x \approx y \wedge y \approx z \Rightarrow x \approx z)$$

$$(g) (\forall x)(\forall y)(\forall z) [(y \approx z) \Rightarrow \\ (x + y \approx x + z) \wedge (y + x \approx z + x)]$$

We want to show that the F.D.S. K_0 is adequate, i.e. we want to prove the following:

Theorem 12: (Adequateness Theorem)

If A is a logically valid formula of a Pred. Logic Λ , then A is provable in K_0 .

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Proposition 6: Let K be a first order F.D.S. and $\neg A$ be a closed formula of K , which is not provable in K . Then the F.D.S. K' obtained by adding A as an axiom to K is consistent.

Proof: Suppose K' is inconsistent. Then for some formula B , $\vdash_{K'} B$ and $\vdash_{K'} \neg B$. We will now give a proof of $\neg A$ in K'

1. $B \Rightarrow (\neg B \Rightarrow \neg A)$ (instance of a tautology)
2. B $\vdash_{K'} B$
3. $\neg B \Rightarrow \neg A$ from 1. & 2. by MP.
4. $\neg B$ $\vdash_{K'} \neg B$
5. $\neg A$ from 3. & 4. by MP

Hence $\vdash_{K'} \neg A$. So $A \vdash_{K'} \neg A$.

Since A is a closed formula, it has no free variables. So by the Deduction theorem

$$\vdash_K A \Rightarrow \neg A$$

$$\text{But } \vdash_K (A \Rightarrow (\neg A)) \Rightarrow (\neg A)$$

because this is an instance of a tautology.

So by MP it follows that $\vdash_K \neg A$.

But this contradicts the fact that $\neg A$ was not provable in K . Hence K' must be consistent.

Theorem 11: If K is a consistent first order F.D.S., then K has a denumerable model, (i.e, there is an interpretation in which all the provable formulas of K are true).

Theorem 12 (A adequateness theorem)

If A is a logically valid formula of K_0 , then A is provable in K_0 .

Proof: First observe that A is a logically valid formula of K_0 iff the closed formula $cl(A)$ is ^{logically} valid (see Property VI) So it will suffice to prove the result for closed formulas.

So let A be a closed formula which is logically. Suppose A is not provable in K_0 . Then by Proposition 5 we can add $\neg A$ as an axiom to K_0 to get a consistent theory K' . Now by Theorem 10, we can find a model of K' , M say.

Then A is true in M because A is logically valid. And $\neg A$ is true in M because $\neg A$ was an axiom of K' and M is a model of K' . But this is clearly a contradiction. $A \& \neg A$ can't both be true in M . Hence A must be provable in K_0 .

Our next task will be to prove Theorem 10. For this we will need several other results.

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Lect. #12 Lemma F : The set of expressions of any first order F.D.S. K is denumerable. Hence the same is true for the set of terms, formulas, and closed formulas.

Proof: Easy. Do for H.W.

Def. A first-order F.D.S. K is said to be complete if, for any closed formula A of K , either $\vdash_K A$ or $\vdash_K \neg A$.

A First order F.D.S. K' is an extension of K if every formula that is provable in K is also provable in K' .

Proposition B : (Lindenbaum's Lemma)

If K is a consistent first order F.D.S., then there is a consistent & complete extension of K .

Proof: From Lemma 6 we know that the set of all closed formulas of K is denumerable. Let B_1, B_2, B_3, \dots be an enumeration of the closed formulas of K . Define a sequence J_0, J_1, J_2, \dots of F.D.S. as follows.

Let $J_0 = K$ and for $n \geq 0$ let

$$J_{n+1} = \begin{cases} J_n \cup \{\neg B_{n+1}\} & \text{if } \neg B_{n+1} \text{ is not provable in } J_n \\ J_n & \text{if } \neg B_{n+1} \text{ is provable in } J_n. \end{cases}$$

Then by Proposition 5, each J is a consistent F.D.S. Let J' be the F.D.S. obtained by using all the axioms of J_0, J_1, J_2, \dots as its set of axioms. Then J' is also consistent.

Indeed, suppose J' was not consistent. Then we can find a formula A such that both A & $\neg A$ are provable in J' . But since a proof involves only a finite no. of axioms, both the proof of A & $\neg A$ can be carried out in J_{n_0} for some large enough n_0 . But this says that J_{n_0} is inconsistent which is a contradiction. Hence J' is consistent.

Clearly J is an extension of K . We will now show that J is complete. Let A be any closed formula of K . Then $A = B_n$ for some n .

But we know that

B_n is provable in J_n or $\neg B_n$ is provable in J_n , from the definition of J_n . So

B_n is provable in J or $\neg B_n$ is provable in J . Hence

either A or $\neg A$ is provable in J . So J is complete.

Def. A closed term is a term which contains no variables.

Examples Let $A = \langle A; f; a, b \rangle$ with A unary & f binary
Then $f(a, b)$ & $f(f(a, a), f(b, b))$ are closed terms.

Def. A first order F.D.S K is called a scape goat F.D.S. if for any formula $A(x)$ with one free variable x , there is a closed term t such that

$$\vdash_K (\exists x) \neg A(x) \Rightarrow \neg A(t).$$

Ex. Let A be the formula $A(x) \wedge \neg A(x)$ where A is a unary Relation symbol. Let

$$L_1 = \{A\} \quad \text{and} \quad L_2 = \{A; a\}$$

Then

$(\exists x) \neg A(x)$ is provable in both $K(L_1)$ & $K(L_2)$. But $K(L_1)$ has no closed terms because L_1 has no constant symbol. Hence $K(L_1)$ is not a scapegoat F.D.S.

However $K(L_2)$ has a chance to be a scapegoat F.D.S. because $\vdash_{K(L_2)} (\exists x) \neg A(x) \Rightarrow \neg A(a)$. (*)

To show that $K(L_2)$ is a scape-goat F.D.S. we will have to show that (*) is true for all formulas A .

Lemma 9: If K is a consistent F.D.S, then we can find a consistent, scapegoat F.D.S. K' which extends K and contains denumerably many closed terms.

Proof: Add to the symbols of K , a denumerable set $\{b_1, b_2, b_3, \dots\}$ of new constant symbols. Call this new theory J_0 . Then J_0 is consistent, because if not, then we would be able to

prove $A \wedge \neg A$ in J_0 for some formula A . But if we replace any b_i appearing in this proof by new variables not appearing in A , then we will get a proof of $A' \wedge \neg A'$ in K (Here A' is just A with the b_i replaced by new variables.) But this contradicts the fact that K is consistent. Hence J_0 is consistent.

Now let $A_1(x_{i_1}), A_2(x_{i_2}), A_3(x_{i_3}), \dots$ be an enumeration of the formulas of J_0 with one free variable. Choose an increasing sequence $j_1 < j_2 < j_3 < \dots$ such that the constant b_{j_n} does not appear in the formulas

$$A_1(x_{i_1}), \dots, A_n(x_{i_n}).$$

Also let

$$S_n = (\exists x_{i_n}) \rightarrow A_n(x_{i_n}) \Rightarrow \neg A_n(b_{j_n}).$$

and let J_n be the F.D.S obtained by adding S_1, \dots, S_n as axioms to J_0 .

Now let J_∞ be the F.D.S obtained by all the S_n 's as axioms to J_0 . Then J_∞ is, by its definition, a scape-goat F.D.S. Since a proof in J_∞ can involve only finitely many of the axioms S_n , it will be a proof in J_{n_0} for a large enough no. So J_∞ will be consistent, once we can show that all the J_n 's are consistent.

We will prove that each J_n is consistent by induction on n . For $n=0$ we know that the result is true.

Now assume that J_{n-1} is consistent. If J_n were inconsistent, then any formula would be provable in J_n . (This is because J_n is a reasonable F.O.S., see Theorem 9, ch. 1). So in particular $\neg S_n$ is provable in J_n i.e., $J_{n-1} \cup \{S_n\} \vdash \neg S_n$. So

$$S_n \vdash_{J_{n-1}} \neg S_n$$

Since S_n is a closed formula, it has no free variables. So by the Deduction theorem

$$\vdash_{J_{n-1}} S_n \Rightarrow \neg S_n$$

But $\vdash_{J_{n-1}} (S_n \Rightarrow \neg S_n) \Rightarrow \neg S_n$ because

$(S_n \Rightarrow \neg S_n) \Rightarrow S_n$ is an instance of a tautology.

So we get $\vdash_{J_{n-1}} \neg S_n$ by MP,

$$\text{i.e. } \vdash_{J_{n-1}} \neg [(\exists x_{in}) \neg A_n(x_{in}) \Rightarrow \neg A_n(b_{jn})]$$

But by conditional Elimination we know

$$\neg(A \Rightarrow B) \vdash_{K_0} A \quad \text{and} \quad \neg(A \Rightarrow B) \vdash_{K_0} \neg B$$

So we get

$$(*) \quad \vdash_{J_{n-1}} (\exists x_{in}) \neg A_n(x_{in}) \quad \text{and} \quad \vdash_{J_{n-1}} \neg \neg A_n(b_{jn})$$

By negation elimination, we also get $\vdash_{J_{n-1}} A_n(b_{jn})$.

Now b_{jn} does not occur in S_1, \dots, S_{n-1} . So we can conclude that $\vdash_{J_{n-1}} A_n(x_p)$ where x_p is any new variable that does not occur in the proof of $A_n(b_{jn})$ in J_{n-1} . By GEN, $\vdash_{J_{n-1}} (\forall x_p) A_n(x_p)$ which contradicts (*). Hence J_n must be consistent & we're done.

Lemma 10: Let J be a consistent, complete, scapegoat F.D.S. Then J has a denumerable model whose domain is the set D of closed terms of J .

Proof: We will first specify an interpretation M with domain D and then show that M is a model of J .

Let $\Lambda(J) = \langle \{R_i\}, \{f_j\}, \{a_k\} \rangle$ be the language of J and $D = \text{set of closed terms of } J$.

1. Let $(a_k)^M = a_k$
2. Specify f_j^M as follows: If $t_1, \dots, t_n \in D$ let $f_j^M(t_1, \dots, t_n) = f_j(t_1, \dots, t_n)$
Note that $f_j(t_1, \dots, t_n)$ is a closed term, so it is an element of D .
3. Also specify R_i^M as follows: If $t_1, \dots, t_n \in D$ let $R_i^M(t_1, \dots, t_n)$ hold iff $t_j R_i(t_1, \dots, t_n)$.

Now to show that M is a model of J , we must show that if B is provable in J , then B is true in M . Since a formula is true in M iff its closure is true in M , it will suffice to prove the result for all closed formulas A of J .

We will prove that for any closed formula A of J

$$(*) \quad \vdash_M A \text{ iff } \vdash_J A$$

by induction on the number r of connectives and quantifiers in A .

Basis: If $r=0$, then A is an atomic formula $R_i(t_1, \dots, t_n)$. And so $(*)$ is a direct consequence of the definition of $(R_i)^M$.

Ind. Step: Suppose the result is true for all formulas with $\leq r-1$ connectives and quantifiers. Let A be a closed formula with r connectives & quantifiers

Case(i): A is $\neg B$

Suppose A is true in M . Then B must be false in M . So by the inductive hypothesis not provable in J . But J is complete.

So $\neg B$ must be provable in J . Hence

$$\vdash_M A \text{ implies } \vdash_J A$$

Now suppose A is false in M . Then B must be true in M . So by the ind. hyp. B must be provable in J . Since J is consistent $\neg B$ cannot be provable in J . Hence

$$\neg(\vdash_M A) \text{ implies } \neg(\vdash_J A)$$

$$\text{So } \vdash_J A \text{ implies } \vdash_M A.$$

Thus we get $(*)$ in case (i).

Case (ii): A is $B \Rightarrow C$.

Suppose A is false in M . Then $\vdash_M B$ and $\vdash_M \neg C$. So by the inductive hyp. B & $\neg C$ are provable in J . So we have

1. $B \Rightarrow (\neg C \Rightarrow \neg(B \Rightarrow C))$ instance of a tautology
2. $\vdash_J B$
3. $\neg C \Rightarrow \neg(B \Rightarrow C)$ 1, 2. & MP
4. $\vdash_J \neg C$
5. $\vdash_J \neg B \Rightarrow C$ 3, 4, & MP

So $\neg A$ is provable in J . Since J is consistent A cannot be provable in J .

$$\therefore \neg(\vdash_M A) \text{ implies } \neg(\vdash_J A)$$

Now suppose A is not provable in J . Then $\neg A$ must be provable in J , because J is complete. So $\vdash_J \neg(B \Rightarrow C)$. But by the Conditional Elimination rules we get

$$\vdash_J B \text{ and } \vdash_J \neg C$$

So by the ind. hyp. B is true in M and C is false in M . $\therefore A = (B \Rightarrow C)$ is false in M . So

$$\neg(\vdash_J A) \text{ implies } \neg(\vdash_M A).$$

Hence (*) is true in this case also.

Case (iii): A is $(\forall x_m)B$ and B is a closed formula.

$$\vdash_M A \text{ iff } \vdash_M \forall x_m B$$

$$\text{iff } \vdash_M B \text{ by Property VI}$$

- iff $\vdash_J B$ by Ind. hyp.
 iff $\vdash_J (\forall x_m) B$ by GEN
 iff $\vdash_J A$.

So (*) is again true in this case.

Case (iv): A is $(\forall x_m) B$ and B is not closed.

Since A is closed, B can only have x_m as a free variable. So $B = F(x_m)$

Suppose $\not\vdash_M A$. Assume that A is not provable in J . Then $\vdash_J \neg A$ because J is complete.

So $\vdash_J \neg(\forall x_m) F(x_m)$. Hence $\vdash_J (\exists x_m) \neg F(x_m)$

Since J is a scapegoat F.D.S., we must have $\vdash_J \neg F(t)$ some closed term t in J .

But $\not\vdash_M A$, so $\not\vdash_M (\forall x_m) F(x_m)$. But by Property X $\not\vdash_M (\forall x_m) F(x_m) \Rightarrow F(t)$ bec.

t is free for x_m in $F(x_m)$. Hence $\not\vdash_M F(t)$

So by the ind. hyp. $\vdash_J F(t)$. But this contradicts the consistency of J . Hence

$\not\vdash_M A$ implies $\vdash_J A$.

Supp. $\vdash_J A$. Assume that A is false in M .

Then $(\forall x_m) F(x_m)$ is false in M . So there is a seq. \underline{s} which does not satisfy $F(x_m)$.

Let t be the m -th component of \underline{s} . Then t is a closed term of J and \underline{s} does not satisfy $F(t)$. So $F(t)$ is false in M . But $\vdash_J A$, i.e.

$\vdash_J (\forall x_m) F(x_m)$. So by the Particularization rule $\vdash_J F(t)$. But by the ind. hyp. $F(t)$ is then true in M , so we have a contradiction. Hence $\vdash_J A$ implies $\not\vdash_M A$. \square

(Denumerable Model Theorem)

Theorem II If K is a consistent first order F.D.S., then K has a denumerable model

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Proof: Let K be a consistent F.D.S. Then we can extend K to a consistent, scape-goat F.D.S K' by Lemma 8. And by Lindenbaum's Lemma, we can extend K' to a complete, consistent first order F.D.S J with the same symbols as K' . Note J will also be a scape-goat F.D.S. Hence by Lemma 9, J has a denumerable model M . Since J is an extension of K , M will also be a model of K .

Corollary 12 (Skolem-Löwenheim theorem)

If a first order F.D.S. K has a model, then it also has a denumerable model.

Proof: Suppose K has a model. Then by Property II, K is consistent. So by Theorem II, K has a denumerable model.

Corollary 13 : (Compactness Theorem for Pred. Logic)

If every finite subset of the axioms in a F.D.S. K has a model, then K has a model.

Proof: Supp. every finite subset of the axioms of K has a model. Then every finite subset of the axioms of K is consistent. Since a proof involves only a finite no. of axioms, there can be no proof of $A \wedge \neg A$ in K . So K is consistent. By theorem II, K has a model.