

Syntax : First we will define the alphabet of the Predicate Logic with Equality (PDL<sup>E</sup>) and then indicate what are the formulas of this logic.

#### A. LOGICAL SYMBOLS

1. variables :  $x_1, x_2, x_3, \dots$
2. primitive connectives :  $\neg, \Rightarrow$
3. primitive quantifier :  $\forall$
4. equality symbol :  $=$
5. Auxiliary symbols
  - a) parentheses :  $(, )$
  - b) comma :  $,$

#### B. NON-LOGICAL SYMBOLS

1. A countable set of relation symbols :  $\{R_i\}$
2. A countable set of function symbols :  $\{f_j\}$
3. A countable set of constant symbols :  $\{c_k\}$

The terms of a PDL<sup>E</sup> are defined recursively in the same way as for a PDL.

1. Variables and constant symbols are terms
2. If  $t_1, \dots, t_n$  are terms and  $f$  is an  $n$ -ary func. symb. then  $f(t_1, \dots, t_n)$  is a term.

The formulas of a PDEL are defined recursively as follows:

1. (a) If  $t_1$  and  $t_2$  are terms then  $(t_1 = t_2)$  is a term formula

(b) If  $t_1, \dots, t_n$  are terms and  $R$  is an  $n$ -ary relation symbol, then  $R(t_1, \dots, t_n)$  is a term formula

These formulas are called atomic formulas

2. If  $A$  and  $B$  are formulas and  $x$  is a variable, then

$(\neg A)$ ,  $(A \Rightarrow B)$ , and  $(\forall x)A$  are also formulas.

Semantics: Now we will say what a formula in a PDEL means.

Let  $\Lambda = \langle \{R_i\}, \{f_j\}, \{a_k\} \rangle$  be the non-logical symbols of a PDEL. A normal interpretation  $M$  of  $\Lambda$  consists of a non-empty domain  $D$  and an assignment function  $\varphi$ .

$\varphi$  assigns an  $n$ -ary relation  $R_i^M \subseteq D^n$  to each  $n$ -ary relation symbol  $R_i$

$\varphi$  assigns an  $n$ -ary function  $f_j^M: D^n \rightarrow D$  to each  $n$ -ary function symbol  $f_j$

$\varphi$  assigns an element  $a_k^M \in D$  to each constant symbol  $a_k$ .

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The equality symbol " $=$ " is, of course, interpreted as the identity relation on  $D$ .

So a normal interpretation

$$M = \langle D, \{R_i^M\}, \{f_j^M\}, \{g_k^M\} \rangle$$

is just a structure which is compatible with  $\Lambda$  in which " $=$ " is interpreted as it should.

A sequence  $\underline{s} = \langle s_1, s_2, s_3, \dots \rangle$  satisfies a formula iff when we replace the free occurrences of each  $x_i$  by  $s_i$  ( $i=1, 2, 3, \dots$ ) the resulting statement is true in  $M$ .

Def. We say that  $\mathcal{A}$  is true in  $M$  (and write  $\models_M \mathcal{A}$ ) if every sequence  $\underline{s} \in \text{SEQ}(M)$  satisfies  $\mathcal{A}$ . We say that  $\mathcal{A}$  is totally false in  $M$  if no seq.  $\underline{s} \in \text{SEQ}(M)$  satisfies  $\mathcal{A}$ .

Let  $\Lambda$  be the non-logical symbols of a PDEL. A formula  $\mathcal{A}$  is said to be logically valid if  $\mathcal{A}$  is true in every interpretation of  $\Lambda$ .

$\mathcal{A}$  is said to be satisfiable if there is an interpretation  $M$  of  $\Lambda$  such that  $\mathcal{A}$  is satisfied by at least one seq.  $\underline{s} \in \text{SEQ}(M)$ .

$\mathcal{A}$  is said to be contradictory if  $\mathcal{A}$  is totally false in every interpretation of  $\Lambda$ .

Def. Let  $\Lambda$  be a PDEL and  $\Gamma$  be a set of formulas of  $\Lambda$ . A normal model of  $\Gamma$  is any normal interpretation of  $\Lambda$  in which every formula of  $\Gamma$  is true.

### A Formal Deductive System for PDEL

$$K_E = \langle \Sigma, \Phi, P, R \rangle$$

We already gave the alphabet & the formulas.

#### P: Axioms :

(A1) - (A5) plus

(A6)  $(\forall x)(x=x)$  for any variable  $x$

(A7)  $(x=y) \Rightarrow (\mathcal{A}(x,x) \Rightarrow \mathcal{A}(x,y))$  for any formula  $\mathcal{A}$ .

Here  $x$  &  $y$  are any variables with  $y$  free for  $x$  in  $\mathcal{A}(x,x)$  and  $\mathcal{A}(x,y)$  is the formula obtained by replace some of the free occurrences of  $x$  by  $y$ 's

#### R: Rules of Inference :

MP : From  $\mathcal{A}$  &  $\mathcal{A} \Rightarrow B$ , infer  $B$

GEN : From  $\mathcal{A}$ , infer  $(\forall x)\mathcal{A}$ .

Lect #15

Proposition 1 : For the F.D.S.  $K_E$  we have(a)  $\vdash_{K_E} (t = t)$  for any term  $t$ (b)  $\vdash_{K_E} (x = y) \Rightarrow (y = x)$ (c)  $\vdash_{K_E} (x = y) \Rightarrow ((y = z) \Rightarrow (x = z))$ Proof(a) 1.  $(\forall x_1) (x_1 = x_1)$  Axiom A62.  $(\forall x_1) (x_1 = x_1) \Rightarrow (t = t)$  Axiom A43.  $(t = t)$  from 1 & 2 by MP(b) 1.  $\vdash (x = x)$  from part (a)2.  $\vdash (x = y) \Rightarrow ((x = x) \Rightarrow (y = x))$  Axiom A73.  $\vdash [(x = y) \Rightarrow ((x = x) \Rightarrow (y = x))] \Rightarrow [(x = x) \Rightarrow ((x = y) \Rightarrow (y = x))]$   
instance of the tautology $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ 4.  $\vdash (x = x) \Rightarrow ((x = y) \Rightarrow (y = x))$  from 2 & 3 by MP5.  $\vdash (x = y) \Rightarrow (y = x)$  from 1 & 4 by MP(c) Let  $A(y, y)$  be  $(y = z)$  &  $A(x, x)$  be  $x = z$ . Then1.  $\vdash (y = x) \Rightarrow ((y = z) \Rightarrow (x = z))$  Axiom A7with  $x, y$  interchanged2.  $\vdash (x = y) \Rightarrow (y = x)$  from part (b)3.  $\vdash ((x = y) \Rightarrow (y = x)) \Rightarrow (((y = x) \Rightarrow (y = z \Rightarrow x = z)) \Rightarrow ((x = y) \Rightarrow (y = z \Rightarrow x = z)))$ instance of the tautology  $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ 4.  $\vdash ((y = x) \Rightarrow (y = z \Rightarrow x = z)) \Rightarrow ((x = y) \Rightarrow (y = z \Rightarrow x = z))$  2, 3, MP5.  $\vdash (x = y) \Rightarrow ((y = z) \Rightarrow (x = z))$  1, 4, MP

Now if we look at the F.D.S.  $K_E$ , we readily see that it is a First order F.D.S. based on the Predicate Logic alone. (Recall that a first-order F.D.S. was just  $K_0$  plus some more axioms.) So we can view the F.D.S. for Predicate plus Equational Logic just a first-order F.D.S. based on the Predicate Calculus.

There is, however, one important thing to remember. In the PDEL the equality symbol " $=$ " is part of the logical symbols and must be interpreted as the identity relation on the domain  $D$ .

In the PDL, the equality symbol " $=$ " is treated as a binary relation symbol, so it is part of the non-logical symbols. Also this binary relation symbol has to satisfy Axioms A6 & A7.

So in any model  $M$  (in the PDL) of  $K_E$  the interpretation of " $=$ " has to be an equivalence relation (because of Prop. 1)

If the interpretation of " $=$ " is the identity relation on  $D$ , then we, of course, say that the model  $M$  is a normal model of  $K_E$ .

Note : Any model  $M$  for  $K_E$  can be contracted to a normal model  $M'$  as follows :

Let the domain of  $M'$  be defined by

$D' = \text{set of all equivalence classes determined by the equiv. rel. } (=)^M \text{ on } D.$

Let  $(a_k)^{M'} = [(a_k)^M]$  ( $\leftarrow$  the equiv. class containing  $(a_k)^M$ )

Let  $(f_j)^{M'}([b_1], \dots, [b_n]) = [f_j^M(b_1, \dots, b_n)]$   
 $=$  the equiv. class containing  $f_j^M(b_1, \dots, b_n)$

And finally, let

$(R_i)^{M'}([b_1], \dots, [b_n])$  be true

iff  $(R_i)^M(b_1, \dots, b_n)$  is true.

Recall that a set  $A$  is denumerable if there is a bijection from  $A$  to  $N$ .

We say that a set  $A$  is countable if  $A$  is finite or if  $A$  is denumerable.

Def. A first order Formal Deductive System with Equality (F.D.S.E) is any F.D.S. in which all of the 7 Axioms of  $K_E$  are provable and which has MP & GEN as rules of inference.

Theorem 2 (Countable Normal model theorem)

Any first order F.D.S.E.  $K$  that is consistent has a countable, <sup>normal</sup> model.

Proof: By Theorem 10 of Ch.2,  $K$  has a denumerable model  $M$ . The contraction of  $M$  yields a normal model  $M'$ . Since  $D$  is denumerable, the number of equiv. classes of  $D$  is finite or denumerable. So  $D'$  is countable. Hence  $M'$  is a countable normal model.

Theorem 3 (Skolem-Lowenheim Thm for PDLE)

Any first order F.D.S.E  $K$  which has an infinite normal model, has a denumerable normal model

Proof: Add to  $K$ , denumerably many new constant symbols  $b_1, b_2, b_3, \dots$  and

$$b_i \neq b_j \quad \text{for all } i \neq j$$

as axioms. Then the resulting F.D.S.E.  $K'$  will be consistent (check this)

By Theorem 12 of Ch.2, we know that  $K'$  has a denumerable model. So as in the proof of Theorem 2 above,  $K'$  will have a countable normal model,  $M'$  say. But since  $K'$  has

$$b_i \neq b_j \quad \text{for all } i \neq j$$

as axioms,  $M'$  is forced to be infinite. Hence  $K'$  has a denumerable model,

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Def. A sentence (in the P.D.L or P.D.L.E) is just a formula which has no free variables (i.e. it is closed)

- Ex. 1.  $(\forall x)(\exists y)(R(x) \wedge \neg R(y))$  is a sentence  
 2.  $(\exists y)(R(y) \Rightarrow R(x))$  is not a sentence.

If  $A$  is a sentence, then  $A$  is either true or false in an interpretation  $M$ .

Note: If  $A$  is a formula which is not a sentence then  $A$  may be neither true nor false in  $M$ .

Ex. Let  $\mathcal{A}$  = "x is prime". Then  $\mathcal{A}$  is satisfied by some seq. in  $M = \langle N, P \rangle$  and is not satisfied by some seq.  
 So  $\mathcal{A}$  is neither true nor false in  $M$ .

Qu. 1 : Is there a satisfiable sentence of the PDL which is false in all interpretations with  $\leq n$  elements?

Aus: Yes.

$$\mathcal{L}_1 = (\exists x)(\exists y)(R(x) \wedge \neg R(y))$$

$\mathcal{L}_1$  is false in all interpretations with  $\leq 1$  element

$$\mathcal{L}_2 = (\exists x)(\exists y)(\exists z)[P(x) \wedge \neg P(y) \wedge Q(y) \wedge \neg Q(z) \wedge R(z) \wedge \neg R(x)]$$

$\mathcal{L}_2$  is false in all interpretations with  $\leq 2$  elements.

Ques 2: Is there a sentence of the PDL which is only true in interpretations with  $\leq n$  elements (or with exactly  $n$  elements)?

Ans: No (and no). If a sentence  $A$  is satisfiable, then by Theorem 12 of our notes in Ch. 2, it will have a denumerable model.

Ques 3: Same as question 2 except that we require the sentence to be in the P.D.L.E and the interpretations to be normal.

Ans: Yes (and yes)

Let  $g_1$  be  $(\exists x)(\forall y)(y=x)$

Then  $g_1$  is true in all normal  $M$ 's with  $\geq 1$  element.

Let  $g_2$  be  $(\exists x)(\exists y)(\forall z)(z=x \vee z=y)$

Then  $g_2$  is true in all normal  $M$ 's with  $\leq 2$  elements

Let  $E_2$  be  $(\exists x)(\exists y)(x \neq y \wedge (\forall z)(z=x \vee z=y))$

Then  $E_2$  is true only in the normal  $M$ 's with exactly 2 elements

Let  $E_3$  be  $(\exists x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge z \neq x \wedge (\forall w)(w=x \vee w=y \vee w=z))$

Then  $E_3$  is true only in the normal  $M$ 's with exactly 3 elements.

Ques 4: Is there a sent. in PDL-E which is true only in int.  $M$ 's?

Q. # 17 Def. In any F.D.S.E. we can introduce a new quantifier  $(\exists! x)$  which is read "there exists exactly one  $x$  such that ..." by defining it as follows.

Let  $(\exists! x) A(x)$  abbreviate

$$((\exists x) A(x)) \wedge (\forall x)(\forall y) [A(x) \wedge A(y) \Rightarrow x = y]$$

Here

the new variable  $y$  is, of course, assumed to be the first variable not occurring in  $A(x)$ .  $(\exists! x)$  is sometimes written as  $(\exists_1 x)$ .

In the P.D.L.E. we can define new functions and get a conservative extension of a theory. This is the essence of the theorem below.

Proposition 5 : Let  $K$  be a first order F.D.S.E. such that  $\vdash_K (\exists! u) A(u, y_1, \dots, y_n)$ . Let

$$\begin{aligned} K' = & K + \text{a new function symbol } f \text{ of arity } n \\ & + (\forall u) [u = f(y_1, \dots, y_n) \Rightarrow A(u, y_1, \dots, y_n)] \\ & + \text{all instances of (A1)-(A7) involving } f. \end{aligned}$$

Then there is a transformation taking each formula  $B$  of  $K'$  to a formula  $B'$  of  $K$  such that

- (a)  $B'$  is  $B$  whenever  $f$  does not occur in  $B$
- (b)  $(\neg B)'$  is  $\neg(B')$
- (c)  $(B \Rightarrow b)'$  is  $B' \Rightarrow b'$
- (d)  $(\forall x B)' is  $(\forall x)(B')$$
- (e)  $\vdash_K B \Leftrightarrow B'$
- (f) If  $\vdash_{K'} B$ , then  $\vdash_K B'$

Hence if  $B$  does not contain  $f$ , then  $\vdash_K B$  implies  $\vdash_K B'$ .

Note: If in  $A(u, y_1, \dots, y_n)$ ,  $n=0$ , then we will be defining constants.

Example: Let  $G$  be the F.D.S.E. with alphabet  $\Lambda = \{+, 0\}$  with axioms: (A1)-(A7) plus.

- $x + (y + z) = (x + y) + z$
- $x + 0 = x$
- $(\forall x)(\exists y)(x + y = 0)$

By using Prop 5 we can eliminate the function symbol "+" and constant "0" and replace them by new relation symbols

The new F.D.S.E.  $G'$  will have alphabet  $\Lambda = \langle P, R \rangle$  where  $P$  is a ternary relation symbol and  $R$  is a unary relation symbol. The relation  $P$  together with the axiom

$$(d) (\forall x_1)(\forall x_2)(\exists! x_3)P(x_1, x_2, x_3)$$

will represent "+". The relation  $R$  together with the axiom

$$(e) (\exists! x_1)R(x_1)$$

will represent "0".

$G'$  will be the F.D.S. consisting of (A1)-(A7) plus (d) & (e) plus

$$(a') P(y, z, w_1) \wedge P(x, w_1, w_2) \wedge P(x, y, w_3) \wedge P(w_3, z, w_4) \Rightarrow (w_2 = w_4)$$

$$(b') R(w_1) \wedge P(x, w_1, w_2) \Rightarrow (w_2 = x)$$

$$(c') (\forall x)(\exists y)(\forall w_1)(\forall w_2) [R(w_1) \wedge P(x, y, w_2) \Rightarrow (w_2 = w_1)]$$

So suppose  $x+x=x$ . By Axiom (c) we can find a  $y$  such that  $x+y=0$ . So

$$\begin{aligned} 0 &= x+y \\ &= (x+x)+y && \text{bec. } x+x=x \\ &= x+(x+y) && \text{by Axiom (a)} \\ &= x+0 && \text{bec. } x+y=0 \\ &= x && \text{by Axiom (b)} \end{aligned}$$

Thus if  $x+x=x$ , then  $x=0$ .

Secondly, we will prove that if  $x_1+x_2=0$  then  $x_2+x_1=0$ . So suppose  $x_1+x_2=0$ . Then

$$\begin{aligned} &(x_2+x_1)+(x_2+x_1) \\ &= ((x_2+x_1)+x_2)+x_1 && \text{by Ax. (a)} \\ &= (x_2+(x_1+x_2))+x_1 && \text{by Ax. (a) again} \\ &= (x_2+0)+x_1 && \text{bec. } x_1+x_2=0 \\ &= (x_2+x_1) && \text{by Ax. (b)} \end{aligned}$$

So  $(x_2+x_1)+(x_2+x_1)=(x_2+x_1)$ . But from the first result we proved we instantly see that we must have  $x_2+x_1=0$ .

$$\therefore x_2+x_2=0 \Rightarrow x_2+x_1=0$$

Now suppose  $x_1+x_3=0$  &  $x_1+x_2=0$ . Then

$$\begin{aligned} x_3 &= 0+x_3 \\ &= (x_2+x_1)+x_3 && \text{bec. } x_1+x_2=0 \\ &= (x_2+x_1)+x_3 && \text{by second fact abv.} \\ &= x_2+(x_1+x_3) && \text{by Axiom (a)} \\ &= x_2+0 = x_2 && \text{by Axiom (b).} \end{aligned}$$

$$\therefore \vdash_G (\exists! x_2)(x_1+x_2=0)$$

Note: If  $n=0$ , (i.e., if there are no  $y_1, \dots, y_n$  in  $A$ ) then we can define a function of 0-variable (i.e., we can define constants).

Example: (Group Theory)

Let  $G$  be the first order F.D.S.E. with alphabet  $\Lambda = \{+, 0\}$  and axioms: (A1)-(A7) plus

- (a)  $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$
- (b)  $x_1 + 0 = x_1$
- (c)  $(\forall x_1)(\exists x_2)(x_1 + x_2 = 0)$

The rules of inference in  $G$  are MP & GEN.

Now we can prove that

$$\vdash_G (\exists ! x_2)(x_1 + x_2 = 0)$$

1.  $(\forall x_1)(\exists x_2)(x_1 + x_2 = 0)$  Axiom (c)
2.  $(\forall x_1)(\exists x_2)(x_1 + x_2 = 0) \Rightarrow (\exists x_2)(x_1 + x_2 = 0)$  Axiom A4
3.  $(\exists x_2)(x_1 + x_2 = 0)$  1, 2 & MP.

We have thus shown that  $(\exists x_2)(x_1 + x_2 = 0)$  is provable in  $G$ . We must now show that  $(\exists ! x_2)(x_1 + x_2 = 0)$

We want to show that if  $x_1 + x_3 = 0$ , then  $x_3 = x_2$ . In order to do this we have to first prove two other results.

First we will show that if  $x + x = x$ , then  $x = 0$

ect. #18

In the Propositional Logic we found that every statement form can be expressed in canonical way known as the Disjunctive Normal Form (D.N.F.)

D.N.F.  $C_1 \vee C_2 \vee \dots \vee C_k$

Each  $C_i$  is an atom .  $i = 1, \dots, k.$

$$C_i = U_{j_1}^{(i)} \wedge U_{j_2}^{(i)} \wedge \dots \wedge U_{j_n}^{(i)}$$

$$U_{j_k}^{(i)} = A_p \text{ or } \neg A_p \quad \xrightarrow{j_k \in \{1, \dots, n\}}$$

Each  $U_{j_k}^{(i)}$  is a literal .

Example : This is in D.N.F.

$$\underbrace{(\neg A_1 \wedge A_2)}_{C_1} \vee \underbrace{(A_1 \wedge A_4)}_{C_2} \vee \underbrace{(\neg A_1 \wedge \neg A_3 \wedge A_4)}_{C_3}$$

In the propositional Logic we also have the following theorem :

Theorem: For any statement form  $\mathcal{A}$ , we can find a statement form  $\mathcal{B}$  in D.N.F. such that

$$\vdash \mathcal{A} \Leftrightarrow \mathcal{B}$$

In the Predicate Logic we have a similar result.

## The Prenex Normal Form

Def. A formula  $(Q_1 y_1)(Q_2 y_2) \dots (Q_n y_n) A$  is said to be in Prenex Normal Form (P.N.F.) if each  $Q_i$  is either  $\forall$  or  $\exists$ , the  $y_i$ 's are all different, and the formula  $A$  contains no quantifiers.

The part  $A$  without the quantifiers is called the matrix of the formula. When  $n=0$ , the formula will contain no quantifiers and we then call it quantifier-free.

Def. A formula  $A(y)$  is said to be similar to  $A(x)$  if  $A(x)$  and  $A(y)$  are the same except that  $A(y)$  has  $y$ 's exactly in the places where  $A(x)$  has free occurrence of  $x$ .

### Examples

1.  $(\exists x) A(x) \Rightarrow B(x, z)$  is sim. to  $(\exists x) A(x) \Rightarrow B(y, z)$
2.  $(\forall z) [A(x) \Rightarrow C(x, z)]$  is sim. to  $(\forall z) [A(y) \Rightarrow C(y, z)]$
3.  $B(x, x) \Rightarrow A(z)$  is not. sim. to  $B(x, y) \Rightarrow A(z)$

We will show that every formula is provably equivalent to one in P.N.F.

Remember Lemma 2.11 says that if  $A(x)$  &  $A(y)$  are similar, then  $\vdash_K (\forall x) A(x) \Leftrightarrow (\forall y) A(y)$

Proposition 6: In any F.D.S.  $K$ , if  $y$  is not free in  $\delta$  and  $b(x) \& b(y)$  are similar, then

- (a)  $\vdash [(\forall x) b(x) \Rightarrow \delta] \Leftrightarrow (\exists y) [b(y) \Rightarrow \delta]$
- (b)  $\vdash [(\exists x) b(x) \Rightarrow \delta] \Leftrightarrow (\forall y) [b(y) \Rightarrow \delta]$
- (c)  $\vdash [\delta \Rightarrow (\forall x) b(x)] \Leftrightarrow (\forall y) [\delta \Rightarrow b(y)]$
- (d)  $\vdash [\delta \Rightarrow (\exists x) b(x)] \Leftrightarrow (\exists y) [\delta \Rightarrow b(y)]$
- (e)  $\vdash \neg(\forall x) b(x) \Leftrightarrow (\exists x) \neg b(x)$
- (f)  $\vdash \neg(\exists x) b(x) \Leftrightarrow (\forall x) \neg b(x)$ .

Proof: See text book p. 83.

Proposition 6 allows us to move interior quantifiers to the front of a formula. This is the essence of the proof of the next result.

Theorem 7: There is an effective procedure for transforming any formula  $A$  into a formula  $B$  in P.N.F. such that  $\vdash A \Leftrightarrow B$ .

Proof: See text book, p. 84.

Example 1: Transform the formula below into an equiv. one in P.N.F.

$$(\forall x)[A(x) \Rightarrow (\forall y)(B(x, y) \Rightarrow \neg(\forall z)C(y, z))]$$

1.  $(\forall x) [ A(x) \Rightarrow (\forall y) (B(x, y) \Rightarrow \neg (\forall z) C(y, z)) ]$   
 $(\forall x) [ A(x) \Rightarrow (\forall y) (B(x, y) \Rightarrow (\exists z) \neg C(y, z)) ]$  by (e)  
 $(\forall x) [ A(x) \Rightarrow (\forall y) (\exists u) (B(x, y) \Rightarrow \neg C(y, u)) ]$  by (d)  
 $(\forall x)(\forall v) [ A(x) \Rightarrow (\exists u) (B(x, v) \Rightarrow \neg C(v, u)) ]$  by (c)  
 $(\forall x)(\forall v)(\exists w) [ A(x) \Rightarrow (B(x, v) \Rightarrow \neg C(v, w)) ]$  by (d)

$(\forall x)(\forall y)(\exists z) [ A(x) \Rightarrow (B(x, y) \Rightarrow \neg C(y, z)) ]$

by changing the bound var.  $v \& w$  to  $y \& z$  resp

2. Ex.2 Transform the formula below into P.N.F.

$$A(x, y) \Rightarrow (\exists y) [ B(y) \Rightarrow ((\exists x) B(x) \Rightarrow C(y)) ]$$

$$A(x, y) \Rightarrow (\exists y) [ B(y) \Rightarrow ((\exists x) B(x) \Rightarrow C(y)) ]$$

$$A(x, y) \Rightarrow (\exists y) [ B(y) \Rightarrow (\forall u) [ B(u) \Rightarrow C(y) ] ]$$
 by (b)

$$A(x, y) \Rightarrow (\exists y)(\forall v) [ B(y) \Rightarrow [ B(v) \Rightarrow C(y) ] ]$$
 by (c)

$$(\exists w) [ A(x, y) \Rightarrow (\forall v) [ B(w) \Rightarrow [ B(v) \Rightarrow C(w) ] ] ]$$
 by (d)

$$(\exists w)(\forall z) [ A(x, y) \Rightarrow (B(w) \Rightarrow [ B(z) \Rightarrow C(w) ] ) ]$$
 by (c)

and we are done.

Note: The main reason for changing to a brand new variable when moving a quantifier is to avoid changing a free occurrence of a variable into a bound occurrence.

### Skolem Normal Form:

A Predicate Logic in which there are no function or constant symbols, and in which there are infinitely many  $n$ -ary relation symbols for each  $n$  is called a Pure Predicate Logic (P.P.L)

Def. A formula of a P.P.L is said to be in Skolem Normal Form if it is of the form

$$(\exists x_1) \dots (\exists x_k)(\forall y_1) \dots (\forall y_m) A$$

where  $A$  is quantifier free.

Theorem 8: For any formula  $A$  in a P.P.L, we can find a formula  $B$  in S.N.F. such that  $\vdash A$  iff  $\vdash B$ .

Example: Convert  $(\forall x)(\exists y) \mathcal{L}(x, y)$  into S.N.F.

Let  $A$  be  $(\forall x)(\exists y) \mathcal{L}(x, y)$ . Then  $\vdash_A A$  iff  $\vdash_{A_1} A_1$ ,

where  $A_1$  is  $(\forall x)[(\exists y) \mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow (\forall x)A(x)$

Now we convert  $A_1$  into P.N.F to get the S.N.F. of  $A$

$$(\forall x)[(\exists y) \mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow (\forall x)A(x)$$

$$(\exists x)[[(\exists y) \mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow (\forall x)A(x)] \quad \text{by 6(a)}$$

$$(\exists x)[(\forall y)[\mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow (\forall x)A(x)] \quad \text{by 6(b)}$$

$$(\exists x)(\exists y)[[\mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow (\forall x)A(x)] \quad \text{by 6(a)}$$

$$(\exists x)(\exists y)(\forall z)[[\mathcal{L}(x, y) \Rightarrow A(x)] \Rightarrow A(z)] \quad \text{by 6(c)}$$

and we are done.

B

Q.L. #19

Ch. 4 - Formal Number Theory

Consider the P.D.L.E with non-logical symbols  $\Lambda = \{ <, s, +, \cdot; 0 \}$ . There are many formulas of  $\Lambda$  which are true in  $(\mathbb{N}, <, s, +, \cdot, 0) = M_0$ , say.

- Examples
1.  $\forall x (x > 0 \rightarrow (\exists y) [s(y) = x])$
  2.  $(\forall x) (\exists x_1) (\exists x_2) (\exists x_3) (\exists x_4) [x_1 \cdot x_1 + x_2 \cdot x_2 + x_3 \cdot x_3 + x_4 \cdot x_4 = x]$

We would like a F.D.S.E  $K$  such that a formula is true in  $M_0$  iff it is provable in  $K$ .

The first attempt at such a Deductive System was made by Dedekind in 1879 and has come to be known as the Peano-Dedekind Axioms.

- (P1) 0 is a natural number
- (P2) If  $x$  is a nat. no., then there is another nat. no.  $x'$  which is called the successor of  $x$
- (P3)  $0 \neq x'$  for any nat. no.  $x$
- (P4) If  $x' = y'$ , then  $x = y$
- (P5) If  $Q$  is a property of nat. nos. such that
  1. 0 has property  $Q$  and
  2. if  $x$  has property  $Q$ , then  $x'$  has property  $Q$
 then all nat. nos. have property  $Q$ .  
 (Principle of Induction)

The Peano-Dedekind Axioms can be used to develop not only Number Theory but also the theory of rational, real, and complex numbers. However a certain amount of set theory is needed. Moreover the Peano-Dedekind axioms do not conform to the constraints a F.D.S.E.

We will give a F.D.S.E  $K_N$  which is often referred to as the Peano F.D.S.E

The non-logical symbols of  $K_N$  are  $\{', +, \cdot, 0\}$ . In addition to the rules MP & GEN,  $K_N$  has

- (A1) - (A5) as axioms plus
  - (S1)  $(x_1 = x_2) \Rightarrow (x_1 = x_3 \Rightarrow x_2 = x_3)$
  - (S2)  $x_1 = x_2 \Rightarrow x_1' = x_2'$
  - (S3)  $0 \neq x_1'$
  - (S4)  $x_1' = x_2' \Rightarrow x_1 = x_2$
  - (S5)  $x_1 + 0 = x_1$
  - (S6)  $x_1 + x_2' = (x_1 + x_2)'$
  - (S7)  $x_1 \cdot 0 = 0$
  - (S8)  $x_1 \cdot x_2' = (x_1 \cdot x_2) + x_1$
  - (S9) For any formula  $A(x)$  of  $K_N$ ,  
 $[A(0) \Rightarrow (A(x) \Rightarrow A(x'))] \Rightarrow (\forall x) A(x)$
- } Equality  
} successor function  
} addition  
} multiplication  
} Induction

Note: Axiom (P5) is not a formalized axiom in the P.D.L.E. Also Axiom P5 says a lot more than Axiom S9. There are  $2^{\aleph_0}$ , i.e. an uncountable no. of properties of nat. nos. — but there are only a countable number of formulas  $A(x)$  in  $K_N$ .

## Models of the F.D.S.E. $K_N$ :

Let  $M_0 = \langle N, S, +, \cdot, 0 \rangle$ . Then it is easy to see that each of the axioms of  $K_N$  is true in  $M_0$ . So  $M_0$  is a model of  $K_N$ . It is called the standard model of  $K_N$ . We will shortly see that there are other models of  $K_N$ .

Def. Since  $0$  is a constant symbol of  $K_N$ , it is a term. So we can successively form the terms

$0, 0', 0'', 0''', \dots$

These terms are called the numerals and we will denote

$\underbrace{0 \cdots}_{n \text{ times}}$  by  $\bar{n}$ .

### Proposition 1

(a) If  $m \neq n$ , then  $\vdash_{K_N} \bar{m} \neq \bar{n}$ .

(b) Any model of  $K_N$  must be infinite.

### Proof:

(a) Suppose  $m \neq n$ . Then  $m < n$  or  $m > n$ .

Case (i) :   $m < n$ . In this case we will show that  $\bar{m} = \bar{n}$  and obtain a contradiction. So it follows that  $\bar{m} \neq \bar{n}$ . All of this has to be done within  $K_N$  however. So here we go

First note that the following formulas are all provable in  $K_N$

$$(S3') \quad 0 \neq t' \quad \text{for any term } t$$

$$(S4') \quad t' = r' \Rightarrow t = r \quad \text{for any terms } t \& r.$$

(see textbook p. 117). Now we give the proof in  $K_N$

$$1. \quad \bar{m} = \bar{n}$$

Hypothesis

$$2. \quad \underbrace{0'''''}_{m \text{ times}} = \underbrace{0'''''}_{n \text{ times}}$$

Expansion of 1.

$$\underbrace{0'''''}_{m-1} = \underbrace{0'''''}_{n-1}$$

by (S4')

$$\underbrace{0'''''}_{m-2} = \underbrace{0'''''}_{n-2}$$

by (S4') again

$$\vdots$$

$$0 = \underbrace{0'''''}_{n-m}$$

by (S4') once more

$$3. \quad 0 = t'$$

where  $t = \overline{(n-m)-1}$

$$4. \quad 0 \neq t'$$

by (S3')

$$5. \quad (0 = t') \wedge (0 \neq t') \quad \text{from 3 \& 4 by Conj. Introd.}$$

$$6. \quad \bar{m} = \bar{n} \vdash_{K_N} (0 = t') \wedge (0 \neq t') \quad \text{steps 1-5.}$$

$$7. \quad \vdash_{K_N} \bar{m} \neq \bar{n}$$

1-6 & Proof by Contradiction

Case (ii) :  $m > n$ . In this case we can again show that  $\bar{m} = \bar{n}$  will lead to a contradiction. So it follow that  $\bar{m} \neq \bar{n}$  again.

Thus if  $m \neq n$ , then  $\vdash_{K_N} \bar{m} \neq \bar{n}$ .

(b) Suppose  $M$  is a model of  $K_N$ . Then  $M$  must have an interpretation for each numeral. But from part (a), there are an infinite number of numerals b/c.

$$\bar{m} \neq \bar{n} \quad (\text{for all } m \neq n.)$$

So  $M$  must be an infinite model.

### Nonstandard models of $K_N$

Def. We can introduce a relation symbol " $<$ " by definition as follows: Let  
 " $t < s$ " abbrev.  $(\exists w)[w \neq 0 \wedge w+t=s]$

Now consider the F.D.S.E.  $K'$  defined by

$$K' = K + \text{a new const. symb. } c$$

$$+ (\bar{n} < c) \quad (\text{for each nat. nos. } n)$$

Then  $K'$  is consistent because any finite subset of  $K'$  has  $\langle N, s, +, \cdot, 0 \rangle$  as a model ( $c$  will be interpreted as a large enough numeral.) So  $K'$  has a normal model  $M$ .

This model  $M$  will also be a model of  $K_N$ .

$M$  cannot be  $M_0$  - because in  $M_0$  there is no nat. nos. which is greater than all the numerals

One possibility for  $M$ .

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots \\
 & 0 & 1 & 2 & \cdots & c-1 & c & c+1 & \cdots & c+c \\
 \xrightarrow{\hspace{1cm}} & \leftarrow & \rightarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow & \cdots \\
 = N + \mathbb{Q} \text{ copies of } \mathbb{Z}.
 \end{array}$$

## Lect. #20 Theorem 6

- (a) A function  $f: \mathbb{N}^n \rightarrow \mathbb{N}$  is representable in  $K_N$  iff  $f$  is recursive.
- (b) A relation  $R \subseteq \mathbb{N}^n$  is expressible in  $K_N$  iff  $R$  is recursive.

### ARITHMETIZATION & GODEL NUMBERS:

Let  $K$  be a first order F.D.S. With each symbol  $u$  in the alphabet of  $K$ , we associate a number  $g(u)$  called the Godel number of  $u$ , in the following way,

$$\begin{array}{lll} g(\epsilon) = 3 & g(\cdot) = 5 & g(\circ) = 7 \\ g(\neg) = 9 & g(\Rightarrow) = 11 & g(\forall) = 13 \end{array}$$

$$P = \{1, 2, 3, \dots\}$$

$$g(f_k^n) = 1 + 8(2^n 3^k) \quad k, n \in P$$

$$g(A_k^n) = 3 + 8(2^n 3^k) \quad k, n \in P$$

$$g(x_k) = 5 + 8(k+1) \quad k \in P$$

$$g(q_k) = 7 + 8k \quad k \in P$$

The Godel number of an expression  $E = u_0 u_1 \dots u_r$  is defined by

$$g(u_0 u_1 \dots u_r) = 2^{g(u_0)} \cdot 3^{g(u_1)} \cdots P_r^{g(u_r)}$$

Here  $P_0 = 2, P_1 = 3, \dots$ , and  $P_r = r\text{-th } \text{odd prime number}$ .

Ex. Let  $E$  be the expression  $A_1^2(x_1, a_2)$ . Then

$$\begin{aligned} g(E) &= 2^{g(A_1^2)} \cdot 3^{g(\cdot)} \cdot 5^{g(x_1)} \cdot 7^{g(\circ)} \cdot 11^{g(a_2)} \cdot 13^{g(\circ)} \\ &= 2^{99} \cdot 3^3 \cdot 5^{21} \cdot 7^7 \cdot 11^{23} \cdot 13^5 \end{aligned}$$

The Gödel number of a sequence of expressions  $E_0, E_1, E_2, \dots, E_r$  is defined by

$$g(E_0, E_1, \dots, E_r) = 2^{g(E_0)} \cdot 3^{g(E_1)} \cdot \dots \cdot p_r^{g(E_r)}$$

Example: Consider the expression:  $a$ ,

If we look at it as a symbol its Gödel no. is 15

If we look at it as an expression its Gödel no. will be  $2^{15}$ . If we look at it as the first expression in a sequence of expressions, its Gödel no. will be  $2^{2^{15}}$ .

The Gödel number of the seq. of expressions:

" $a_1$ ", " $A'_1(a_1)$ ", " $x_1$ "

is  $2^{g(a_1)} \cdot 3^{g(A'_1(a_1))} \cdot 5^{g(x_1)}$

$$= 2^{2^{23}} \cdot 3^{2^{51} \cdot 3^3 \cdot 5^{15} \cdot 7^5} \cdot 5^{2^{21}}$$

Note:

1. The Gödel number of a symbol is always odd  
 If it is  $\geq 13$  &  $\equiv 1 \pmod{8}$  it is a function symbol  
 If it is  $\geq 13$  &  $\equiv 3 \pmod{8}$  it is a relation symbol  
 If it is  $\geq 13$  &  $\equiv 5 \pmod{8}$  it is a constant symbol  
 If it is  $\geq 13$  &  $\equiv 7 \pmod{8}$  it is a variable
2. In a single expr., the power of 2 is always odd
3. In a seq. of expr., the power of 2 is always even.

Proposition 7 (see Prop. 3.25 in textbook) <sup>& 3.26 3.27</sup>

Let  $K$  be a first order F.D.S. such that the relations

- (a)  $IC(x) \Leftrightarrow x$  is the Gödel no. of a const. symb.
- (b)  $FL(x) \Leftrightarrow x$  is the Gödel no. of a func. symb.
- (c)  $PL(x) \Leftrightarrow x$  is the Gödel no. of a relation symb.

are all recursive. Then so are the following 16 relations & functions

#1.  $EVbl(x) \Leftrightarrow x$  is the Gödel no. of an expr. which just a variable.

$$EVbl(x) \Leftrightarrow (\exists z)(z < x \wedge 1 \leq z \wedge x = 2^{5+8(z+1)})$$

#2.

.

#14  $\text{Neg}(x)$  = the function which produces the Gödel no. of  $(\neg A)$ , when  $x$  is the Gödel no. of  $A$

#15  $\text{Cond}(x, y)$  = the function which produces the Gödel no. of  $(A \Rightarrow B)$  when  $x$  &  $y$  are the Gödel nos. of  $A$  &  $B$  resp.

#16  $\text{Clos}(u)$  = the function which produces the Gödel number of closure( $A$ ) when  $x$  is the Gödel no. of  $A$ .

If in addition  $K$  contains the individual constant  $0$  and the function symbol  $f'_1$  of  $K_N$  and

(d)  $\text{PrAx}(y) \Leftrightarrow y$  is the Gödel no. of a <sup>proper</sup> axiom of  $K$  is rel.,  
then so are the following functions & relations

#17  $\text{Num}(y) =$  the Gödel no. of the numeral  $\bar{y}$

#18  $\text{Nu}(x) \Leftrightarrow x$  is the Gödel no. of a numeral

#19  $\text{Du}(M) =$  the Gödel no. of  $A(\bar{u})$  when  $u$   
is the Gödel no. of  $A(x_1)$

#20  $\text{Ax}(y) \Leftrightarrow y$  is the Gödel no. of an  
axiom of  $K$

#21  $\text{Prf}(y) \Leftrightarrow y$  is the Gödel no. of a  
proof in  $K$

#22  $\text{Pf}(y, x) \Leftrightarrow y$  is the Gödel no. of a proof  
of the formula whose Gödel  
number is  $x$ .

For the proofs, see the textbook p. 150 - 156.

The proof of theorem 6 uses this arithmetization  
of the Predicate Logic. See Prop. 3.28 &  
Cor. 3.29 on p. 156 & 157.

Lect. #21 Recall that the function  $D(u)$ , defined by

$D(u) = \text{the Godel no. of } A(\bar{u}) \text{ where } u \text{ is}$   
 $\text{the Godel no. of } A(x_1) \quad (\text{see } \#19)$

Def. If  $\ell$  is a formula and  $u = \text{Godel number of } \ell$ ,  
we will use  $[\ell]$  to denote  $\bar{u}$ .

Proposition 8 : (Diagonalization Lemma)

Let  $K$  be a F.D.S. with the same alphabet as  $K_N$   
such that  $D(u)$  is representable in  $K$ . Then for any  
formula  $B(x_1)$  we can find a closed formula  $\ell$   
such that  $\vdash_K \ell \Leftrightarrow B([\ell])$ .

Proof. Let  $\mathcal{D}(x_1, x_2)$  be a formula which represents  
 $D$  in  $K$  and consider the formula

$$(V) \quad (\forall x_2) [\mathcal{D}(x_1, x_2) \Rightarrow B(x_2)]$$

Let  $m$  be the Godel no. of  $(V)$  and substitute  
 $\bar{m}$  for  $x_1$  in  $(V)$  to get the formula

$$(b) \quad (\forall x_2) [\mathcal{D}(\bar{m}, x_2) \Rightarrow B(x_2)]$$

Now let  $q$  be the Godel no. of  $\ell$ . Then  
 $\bar{q} = [\ell]$  and  $D(m) = q$  (from the def. of  $D$ ).

Since  $\mathcal{D}(x_1, x_2)$  is a formula which represents  $D$

$$(d) \quad \vdash_K \mathcal{D}(\bar{m}, \bar{q}) \& \vdash_K (\exists x_2) \mathcal{D}(\bar{m}, x_2)$$

From  $(d)$  &  $(b)$ , it is not difficult to see that  
 $\vdash_K \ell \Leftrightarrow B(\bar{q})$ . So  $\vdash_K \ell \Leftrightarrow B([\ell])$   
(see p. 160 for details)

Def. A sentence  $A$  of an F.D.S.  $K$  is said to be undecidable in  $K$  if neither  $A$  nor  $\neg A$  is provable in  $K$ .

We want to find an undecidable sentence of  $K_N$ . This will show that the F.D.S.  $K_N$  is not complete.

Let  $K$  be an F.D.S. with the same alphabet as  $K_N$  such that # $A$ . The relation  $\text{PrAx}(y)$  is recursive

#B. It is provable in  $K$  that  $0 \neq 0'$

#C. Every recursive function is representable in  $K$ .

Then we can find a sentence  $g$  of  $K$  which is undecidable in  $K$ .

Recall that the relation

$\text{Pf}(y, x) \Leftrightarrow y$  is the Godel no. of a proof in  $K$  of the formula with Godel no.  $x$ ,

was recursive.

(see #22 p.155)

Since all recursive relations are expressible in  $K$ , we can find a formula  $\text{Pf}(x_2, x_1)$  which shows that  $\text{Pf}$  is expressible in  $K$ .

Let  $B(x_1)$  be the formula  $(\forall x_2)[\neg \text{Pf}(x_2, x_1)]$ .

Then by Prop. 8 we can find a sentence  $g$  such that  $\vdash g \Leftrightarrow B(g^7)$ ,

i.e.,  $\vdash_K g \Leftrightarrow (\forall x_2)[\neg \text{Pf}(x_2, g^7)]$

This sentence  $g$  is called a Godel sentence of  $K$ .

Note that  $g$  says that "for all  $x_2$ ,  $x_2$  is not a proof in  $K$  of  $g$ ", i.e. " $g$  is not provable in  $K$ ". So  $g$  intuitively says "I am not provable in  $K$ "

### Theorem 10 (Gödel's first incompleteness theorem)

Let  $K$  be an F.D.S. which satisfies  $\#A, \#B \& \#C$ .  
Then we have

- (a) if  $K$  is consistent, then  $\bar{g}$  is not provable in  $K$
- (b) if  $K$  is  $\omega$ -consistent, then  $\neg\bar{g}$  is not provable in  $K$

Proof:

(a) Suppose  $\bar{g}$  is provable in  $K$ . Let  $q = \text{Gödel no. of } \bar{g}$  and  $r = \text{Gödel no. of a proof of } \bar{g} \text{ in } K$ . Then  $\text{Pf}(r, q)$  is true. Hence  $\vdash_K \text{Pf}(\bar{r}, \bar{g})$ , i.e.  $\vdash_K \text{Pf}(\bar{r}, \bar{g}^7)$ . But  $\vdash_K \bar{g} \Leftrightarrow (\forall x_2)[\neg \text{Pf}(x_2, \bar{g}^7)]$ . Since  $\bar{g}$  is provable in  $K$ , it follows that  $\vdash_K (\forall x_2)(\neg \text{Pf}(x_2, \bar{g}^7))$ . By the Particularization Rule it follows that  $\vdash_K \neg \text{Pf}(\bar{r}, \bar{g}^7)$  which together with  $\vdash_K \text{Pf}(\bar{r}, \bar{g}^7)$  makes  $K$  inconsistent. Hence  $\bar{g}$  is not provable in  $K$ .

(b) Suppose  $K$  is  $\omega$ -consistent. Assume that  $\bar{g}$  is provable in  $K$ . Since  $K$  is  $\omega$ -consistent,  $K$  is consistent and since  $\neg\bar{g}$  is provable in  $K$ ,  $\neg\bar{g}$  is true in all models of  $K$ . So  $\bar{g}$  cannot be provable in  $K$  (otherwise we would get that both  $\neg\bar{g}$  &  $\bar{g}$  are true in a model of  $K$  which is impossible).

Since  $\vdash_K \bar{g} \Leftrightarrow (\forall x_2)\neg \text{Pf}(x_2, \bar{g}^7)$  &  $\vdash_K \neg\bar{g}$  we get  $\vdash_K \neg(\forall x_2)\neg \text{Pf}(x_2, \bar{g}^7)$ . So  $\vdash_K (\exists x_2)\text{Pf}(x_2, \bar{g}^7)$ . Since  $\bar{g}$  is not provable in  $K$ ,  $\text{Pf}(n, q)$  is false for each  $n \in \mathbb{N}$ . So  $\vdash_K \neg \text{Pf}(\bar{n}, \bar{g}^7)$  for each  $n \in \mathbb{N}$ .

(Remember  $\bar{g} = \bar{g}^7$ ). Since  $K$  is  $\omega$ -consistent we get that  $(\exists x_2)\text{Pf}(x_2, \bar{g}^7)$  is not provable in  $K$  - contradicting  $\vdash_K \exists x_2 \text{Pf}(x_2, \bar{g}^7)$ . Hence  $\neg\bar{g}$  is not provable in  $K$ .

(91)

Corollary 11: In the F.D.S.  $K_N$ ,  $\mathbf{g}$  is an undecidable sentence.

Proof: Just observe that  $K_N$  is  $\omega$ -consistent and that  $\#A$ ,  $\#B$  &  $\#C$  are satisfied in  $K_N$ . Then apply Theorem 10.

The sentence  $\mathbf{g}$  intuitively says "I am unprovable in  $K_N$ ". Because of this we can give a rough heuristic argument which will show why  $\mathbf{g}$  is undecidable. We know  $K_N$  is consiste.

- (a) Suppose  $\mathbf{g}$  was provable in  $K_N$ . Then  $\mathbf{g}$  would be true. (This follows because  $K_N$  is consistent) But  $\mathbf{g}$  says that "I am not provable in  $K_N$ ", and since  $\mathbf{g}$  is true, it follows that  $\mathbf{g}$  is not provable in  $K_N$ . This is a contradiction. So  $\mathbf{g}$  cannot be provable in  $K_N$ .
- (b) Suppose  $\neg\mathbf{g}$  was provable in  $K_N$ . Then  $\neg\mathbf{g}$  would be true (bec.  $K_N$  is consistent).  $\mathbf{g}$  must be false. But  $\mathbf{g}$  says that "I am not provable in  $K_N$ " and since  $\mathbf{g}$  is false, it follows that  $\mathbf{g}$  is provable in  $K_N$ . But  $\neg\mathbf{g}$  and  $\mathbf{g}$  can't be both provable in  $K_N$  because  $K_N$  is consistent. Hence we have a contradiction. So  $\neg\mathbf{g}$  cannot be provable in  $K_N$ .

Note: This "proof" is not rigorous. So it is not really a proof.

Theorem 13 (Godel's 2nd incompleteness theorem)

If  $K$  is consistent F.D.S. which contains  $KN$ , then  $\mathcal{C}_K$  is not provable in  $K$ .

Proof: We know that if  $K$  is consistent, then  $\mathcal{G}$  is not provable in  $K$ . So the sentence

$$\mathcal{C}_K \Rightarrow \mathcal{G}$$

is true (i.e. we have a proof in ordinary mathematics)

Now it can be shown that this ordinary proof can be converted to a formal proof in  $K$ .

So we actually have that

$$\mathcal{C}_K \Rightarrow \mathcal{G}$$

is provable in  $K$ .

So if  $K$  is consistent, then  $\mathcal{C}_K$  cannot be provable in  $K$ .

(Otherwise, we get  $\vdash_K \mathcal{C}_K$

$$\vdash_K \mathcal{C}_K \Rightarrow \mathcal{G}$$

so  $\vdash_K \mathcal{G}$  by MP.

which is impossible by Theorem 10(a) )

Hence  $\mathcal{C}_K$  is not provable in  $K$  if  $K$  is consistent

Lec #22

We would to find a sentence which is undecidable whenever the F.D.S.  $K$  is consistent. In Theorem 10, we need to know that  $K$  is  $\omega$ -consistent before we can conclude that  $\mathcal{L}$  is undecidable in  $K$ .

Recall that the function

$\text{Neg}(x) = \text{The Godel number of } \neg A \text{ where } A \text{ is the formula with Godel no. } x \text{ is recursive.}$  see #14 p.134.

Now let  $K$  be an F.D.S. which satisfies  $\#A$ ,  $\#B \& \#C$  and also

$$\#D \quad \vdash_K (x \leq \bar{n}) \Rightarrow (x=0 \vee x=1 \vee \dots \vee x=\bar{n}) \quad \text{for each } n \in \mathbb{N}$$

$$\#E \quad \vdash_K (x \leq \bar{n}) \vee (\bar{n} \leq x) \quad \text{for each } n \in \mathbb{N}.$$

Then all recursive functions will be representable in  $K$ .

Let  $N(x_2, x_1)$  be a formula which represents the function  $\text{Neg}$  in  $K$  and let  $E(x_1)$  be the formula

$$(\forall x_2) \left( P_f(x_2, x_1) \Rightarrow (\forall x_3) [ N(x_2, x_1) \Rightarrow (\exists x_4) (x_4 \leq x_2 \wedge P_f(x_4, x_3)) ] \right),$$

By the Diagonalization theorem, we can find a sentence  $R$  such that

$$\vdash_K R \Leftrightarrow E(R')$$

Such a sentence is called a Rosser sentence

Theorem 12 (Rosser's extension of Gödel's theorem)

Let  $K$  be a F.D.S. which satisfies  $\#A, \#B, \#C, \#D, \& \#E$ . If  $K$  is consistent, then  $R$  is an undecidable sentence in  $K$ .

Proof: see textbook p. 161.

Intuitively, the sentence  $R$  says that

"If  $R$  has a proof in  $K$ , with Gödel no.  $x_2$ ,  
then  $\neg R$  has a proof in  $K$  with Gödel  
number smaller than  $x_2$ "

This is a roundabout way of saying

"Under the assumption of consistency, I  
am unprovable in  $K$ "

Let  $\mathcal{G}_K$  be the formula

$$(\forall x_1)(\forall x_2)(\forall x_3)(\forall x_4) \neg [Pf(x_1, x_3) \wedge Pf(x_2, x_4) \wedge N(x_3, x_4)]$$

$\mathcal{G}_K$  says that "there is no proof of both a formula  
and its negation in  $K$ "

This is saying that " $K$  is consistent."

We will next show that  $\mathcal{G}_K$  is not provable in  $K$   
for any strong enough  $K$ .

Theorem 13 (Gödel's 2nd incompleteness theorem)

If  $K$  is consistent F.D.S. which contains  $K_N$ ,  
then  $\mathbb{G}_K$  is not provable in  $K$ .

Proof: We know that if  $K$  is consistent,  
then  $\mathbb{G}$  is not provable in  $K$ . So the  
sentence

$$\mathbb{G}_K \Rightarrow \mathbb{G}$$

is true (i.e. we have a proof in ordinary mathematics).

Now it can be shown that this ordinary proof  
can be converted to a formal proof in  $K$ .  
So we actually have that

$$\mathbb{G}_K \Rightarrow \mathbb{G}$$

is provable in  $K$ .

So if  $K$  is consistent, then  $\mathbb{G}_K$  cannot  
be provable in  $K$ .

(Otherwise, we get  $\vdash_K \mathbb{G}_K$

$$\vdash_K \mathbb{G}_K \Rightarrow \mathbb{G}$$

so  $\vdash_K \mathbb{G}$  by MP.

which is impossible by Theorem 10(a) )

Hence  $\mathbb{G}_K$  is not provable in  $K$  if  
 $K$  is consistent