

Lect #1 Introduction to Mathematical Logic

Qu: What is a logic?

Ans: A logic is a systematic study of a certain kind of valid inferences

Qu: What is Mathematical Logic?

Ans: Mathematical Logic is the study of the logics that are used in mathematics by using the methods (techniques) of mathematics

Qu: Why do we study mathematical logic?

- Ans:
1. In order that we may resolve certain paradoxes
  2. In order that we may understand the scope and limitation of mathematics.

Paradoxes:

1. Zeno's paradox:



2. Epimedes paradox:

"This sentence is false"

3. Barber's paradox:

"A certain barber shaves exactly those men who do not shave themselves. Does the barber shave himself?"

4. Zermelo-Russell's paradox: "Let  $R = \{x : x \notin x\}$ . Is  $R \in R$ ?"

Mathematics is considered the most exact of sciences (and it is!) Why is mathematics so exact? It is because of the axiomatic method. ②

The Axiomatic method (as used in, say, Euclidean geometry, (Formal Deductive System))

1. Axioms

(a) Logical Axioms :  $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$  ,  $\neg(P \vee Q) \Rightarrow \neg P \wedge \neg Q$

(b) Mathematical Axioms (Postulates)

"Two non-parallel lines intersect in exactly one point."

2. Proofs

Sequence of statements  $s_1, s_2, \dots, s_n$  such that each  $s_i$  is an axiom or a direct consequence of some of the preceding statements  $s_1, \dots, s_{i-1}$ .

3. Theorems :

Any statement that can be deduced from the axioms by valid inference. ~~The~~ last statement in a proof is always a theorem.

Any branch of mathematics (Euclidean Geometry, Group Theory, Real Analysis, Set Theory, ... etc) can be cast into a Formal Deductive system.

We can even cast Logics such as Propositional Logic, Predicate Logic, etc. into Formal Deductive systems.

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The main focus of this course is to show the power & limitation of the Axiomatic Method.

Suppose want to <sup>Speak,</sup> Sanskrit.

1. Need to know the alphabet & phonetics - syntax
2. Need to know the meaning of sanskrit words - semantics

Sanskrit is called the object language. The language in which we try to ~~understand~~ <sup>communicate</sup> Sanskrit will be English - and is called the meta-language.

### Ch. I - Propositional Logic.

1. ~~Language~~ of Propositional Logic :  $(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$
2. Semantics of Propositional Logic :  
logically valid, logically implies, logically equivalent
3. An algorithm: for answering questions <sup>in</sup> Prop. Logic
4. A Formal Deductive System for Prop. Logic.  $\mathcal{L}_0$

### Ch. II - Predicate Logic

1. Language of Predicate Logic
2. Semantics of Predicate Logic  
Structures, interpretations, valid formulas, models
3. No algorithmic method for answering questions
4. Need a Formal Deductive System for Pred. Logic  $\mathcal{K}_0$
5. System is sound & complete  
(Gödel's completeness theorem)

Ch. III - Predicate Logic With Equality

- 1. Language of Pred. Logic with Eq.
- 2. Semantics of " " " " " "  
Normal models
- 3. No algorithmic method
- 4. A Formal Deductive System KE
- 5. System is sound & complete.

Ch. IV - Formal Number Theory

- 1. Lang. & Semantics of Number Theory
- 2. Models of Number theory
- 3. A Formal Deductive System KN
- 4. System is sound but not complete for Th( $M_0$ )
- 5. There is no recursive axiomatization of number theory which is complete. } Godel's 1st Incompleteness Theorem
- 6. Godel's second incompleteness Theorem

In Any sufficiently strong theory (such as Number Theory & Set Theory) the statement which says that the Theory is consistent is not provable in that theory.

A primitive proposition is a declarative sentence. It can be true or false. By using sentential connectives we can form complex propositions. Propositional logic is the study of the relationships between complex propositions and the relationships between these propositions.

### The syntax of Propositional Logic:

The alphabet of PPL consists of

1. statement letters :  $A, B, C, \dots, Z, A_1, A_2, A_3, \dots$

2. connectives.

$\neg$  (negation)  $\Rightarrow$  (conditional)

$\wedge$  (conjunction)  $\Leftrightarrow$  (biconditional)

$\vee$  (disjunction)

3. parentheses :  $(, )$

A statement form is any expression that is built from the statement letters in the appropriate manner. To be more precise we define a statement forms recursively as follows:

1. All statement letters are statement forms

2. If  $A$  &  $B$  are statement forms then so are  $(\neg A)$ ,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \Rightarrow B)$ , and  $(A \Leftrightarrow B)$ .

3. Only the expressions that can be formed from things in 1. by using the rules in 2. will be statement forms

## Examples of statement forms :

1.  $((\neg A) \vee B) \Rightarrow (A \Leftrightarrow B)$
2.  $((A \wedge (A \Rightarrow B)) \Rightarrow B)$
3.  $((A \wedge \neg A) \Rightarrow (A \Rightarrow B))$

Qu: What does a statement form mean?

## The semantics of Propositional Logic

The statement forms become true or false when we inject truth-values to the statement letters and clarify the truth-functional character of each of the connectives.

Meaning of " $\neg$ " :

A	$(\neg A)$
T	F
F	T

Meanings of " $\wedge, \vee, \Rightarrow, \Leftrightarrow$ " :

A	B	$(A \wedge B)$	$(A \vee B)$	$(A \Rightarrow B)$	$(A \Leftrightarrow B)$
T	T	T	T	T	T
F	T	F	T	T	F
T	F	F	T	F	F
F	F	F	F	T	T

$(A \wedge B)$  is read as "A and B"

$(A \vee B)$  " " "A or B"

$(A \Rightarrow B)$  " " "A suffices for B"

$(A \Leftrightarrow B)$  " " "A exactly when B"

$(\neg A)$  " " "Not A"

So if we are given the truth values of the constituent statement letters of a statement form, we can find out if the statement form is true or false.

Def. A statement form  $A$  is said to be logically valid <sup>(or a tautology)</sup> if it is always true no matter what the truth values of its statement letters may be.

A statement form  $A$  is said to be satisfiable if it is true for some assignment of truth values to its constituent statement forms.

A statement form  $A$  is said to be a contradiction if it is not satisfiable.

Examples

1.  $(A \wedge (A \Rightarrow B)) \Rightarrow B$  is a logically valid st. form

A	B	$(A \wedge (A \Rightarrow B))$			$\Rightarrow$	B
T	T	T	T	T	T	T
F	T	F	F	T	T	T
T	F	T	F	F	T	F
F	F	F	F	T	T	F

2.  $(A \wedge B) \Rightarrow (\neg A)$  is satisfiable

A	B	$(A \wedge B)$			$\Rightarrow$	$(\neg A)$
T	T	T	T	T	F	F
F	T	F	T	F	T	T
:	:	:	:	:	:	:

stop yes!

3.  $A \wedge (\neg A)$  is a contradiction

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Def. We say that  $A$  logically implies  $B$  if the statement form  $(A \Rightarrow B)$  is logically valid.

We say that  $A$  is logically equivalent to  $B$  if the statement form  $(A \Leftrightarrow B)$  is logically valid.

Examples:

1.  $A \wedge (A \Rightarrow B)$  logically implies  $B$   
because  $(A \wedge (A \Rightarrow B)) \Rightarrow B$  is logically valid

2.  $(\neg A) \vee B$  is logically equivalent to  $A \Rightarrow B$   
because  $(\neg A \vee B) \Leftrightarrow (A \Rightarrow B)$  is logically valid

A	B	$(\neg A \vee B) \Leftrightarrow (A \Rightarrow B)$				
T	T	F	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	F	T	F
F	F	T	F	F	T	T

Fact: By using a truth table we can answer any question in Prop. Log. algorithmically.

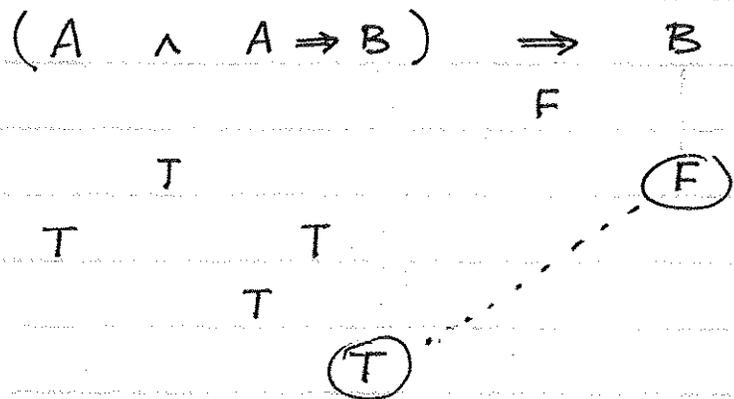
Questions

1. Is  $A$  logically valid?
2. Is  $A$  satisfiable?
3. Is  $A$  a contradiction?
4. Does  $A$  logically imply  $B$ ?
5. Is  $A$  logically equiv. to  $B$ ?

Lect #3 The Short-cut Method

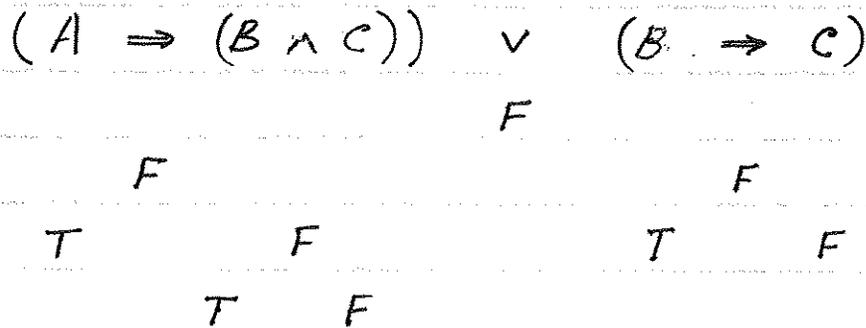
In Propositional Logic most of the questions boils down to checking whether or not a st. form is a tautology or is satisfiable. Unfortunately the method of using truth tables is rather slow and cumbersome. Below we indicate a ~~shortcut~~ method which is shorter sometimes

Ex.1 Is  $(A \wedge A \Rightarrow B) \Rightarrow B$  a tautology



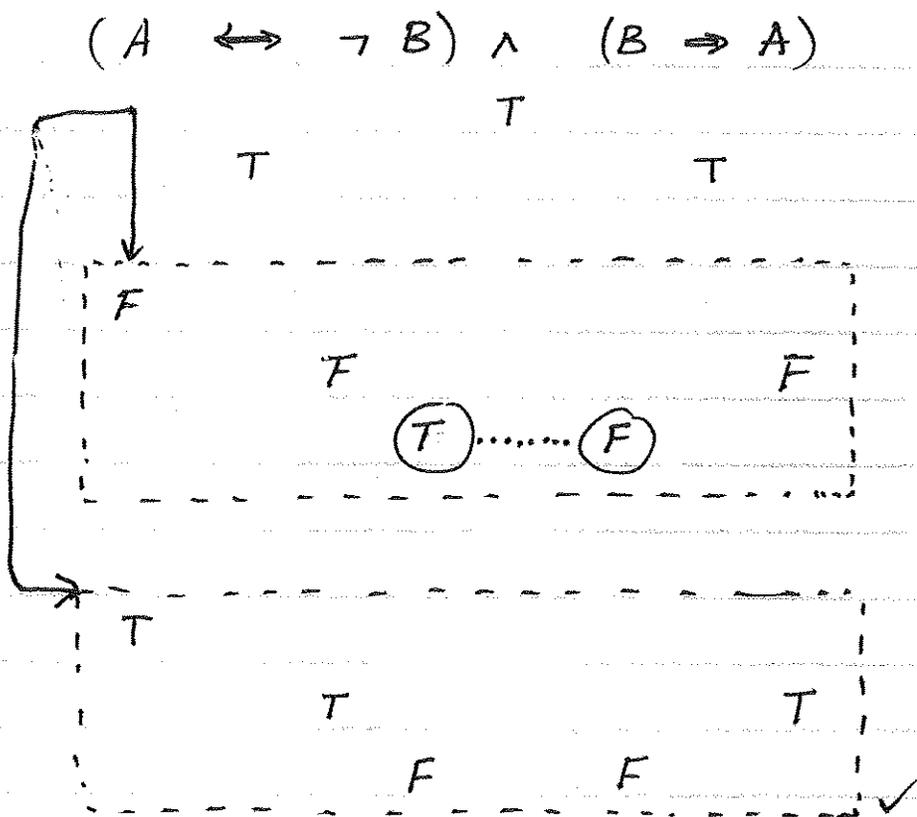
So it is impossible for  $(A \wedge A \Rightarrow B) \Rightarrow B$  to be false. Hence  $(A \wedge A \Rightarrow B) \Rightarrow B$  is a tautology

Ex.2 Is  $(A \Rightarrow (B \wedge C)) \vee (B \Rightarrow C)$  a tautology?



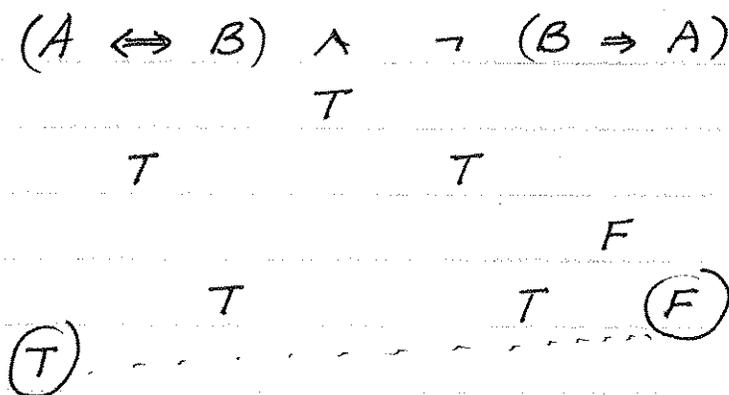
So it is possible for  $(A \Rightarrow (B \wedge C)) \vee (B \Rightarrow C)$  to be false. Hence it is not a tautology.

Ex.3 Is  $(A \Leftrightarrow \neg B) \wedge (B \Rightarrow A)$  satisfiable?



So  $(A \Leftrightarrow \neg B) \wedge (B \Rightarrow A)$  is satisfiable.

Ex.4 Is  $(A \Leftrightarrow B) \wedge \neg(B \Rightarrow A)$  satisfiable?



Hence it is impossible for  $(A \Leftrightarrow B) \wedge \neg(B \Rightarrow A)$

# Adequate sets of connectives

Def. A truth-function of  $n$  variables is any function  $f: \{T, F\}^n \rightarrow \{T, F\}$

The meanings of each of the connectives  $\neg, \wedge, \vee, \Rightarrow$  and  $\Leftrightarrow$  were specified by truth-functions.

Qu: How many truth functions of  $n$  variables are there?

1 variable :

A	$\neg(A)$	$\neg\neg(A)$	$\neg\neg\neg(A)$
T	F	T	F
F	T	F	T

2 variables :

A	B	$\neg A$	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$	$A \downarrow B$	$A \uparrow B$
T	T	F	F	T	T	T	T	F	F
T	F	F	T	F	T	F	F	F	T
F	T	T	F	F	T	T	F	F	T
F	F	T	T	F	F	T	T	T	T

Ans: There are  $2^{2^n}$  truth-functions of  $n$  variables.

$A \wedge B$	$\neg(A \Rightarrow B)$	$\neg(A \Leftrightarrow B)$	$\neg A \wedge B$	$A \vee \neg B$
F	F	F	F	T
F	T	T	T	F
T	T	F	F	T
F	F	F	F	T

Def. Let  $A_1, \dots, A_n$  be  $n$  truth-functional variables. An expression of the form

$$A_i \text{ or } \neg A_i$$

is said to be a literal.

An expression of the form  $U_1 \wedge U_2 \wedge U_3 \wedge \dots \wedge U_n$  where  $U_i$  either  $A_i$  or  $\neg A_i$  is said to be an atom.

Theorem 1: Any truth function of  $n$  variables can be expressed as a disjunction of atoms. In other words any stat. form can be expressed in terms of  $\neg, \wedge$  and  $\vee$ .

Ex.

A	B	C	$f(A, B, C)$
T	T	T	F
F	T	T	T
T	F	T	F
F	F	T	F
T	T	F	T
F	T	F	F
T	F	F	T
F	F	F	F

$$\neg A \wedge B \wedge C$$

$$A \wedge B \wedge \neg C$$

$$A \wedge \neg B \wedge \neg C$$

$$f(A, B, C) = ((\neg A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C)) \vee (A \wedge \neg B \wedge \neg C)$$

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Corollary 2: Every stat. form can be expressed in terms of  $\neg$  and  $\wedge$ .

Proof: In view of Thm. 1 it will suffice to show that " $\vee$ " can be expressed in terms of  $\neg$  and  $\wedge$ . But we know that

$A_1 \vee A_2 \vee \dots \vee A_n$  is log. equiv. to  $\neg(\neg A_1 \wedge \neg A_2 \wedge \dots \wedge \neg A_n)$   
So we are done.

Ex.  $f(A, B, C) = (\neg A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee A \wedge \neg B \wedge \neg C$

$$f(A, B, C) = \neg[\neg(\neg A \wedge B \wedge C) \wedge \neg(A \wedge B \wedge \neg C) \wedge \neg(A \wedge \neg B \wedge \neg C)]$$

Def. A set of connectives is said to be adequate if all stat. forms can be expressed by using only connectives from that set.

Ex.  $\{\neg, \wedge, \vee\}$  and  $\{\neg, \wedge\}$  are adequate.

Show that  $\{\neg, \vee\}$  and  $\{\neg, \Rightarrow\}$  are adequate.

Prop. 3:  $\{\downarrow\}$  and  $\{\uparrow\}$  are adequate sets

$A \downarrow B$  means  $\neg(A \vee B)$

So  $A \downarrow A$  is equiv. to  $\neg(A \vee A)$  which is equiv. to  $\neg A$

Also  $(A \downarrow A) \downarrow (B \downarrow B)$  is equiv. to  $(\neg A) \downarrow (\neg B)$

which is equiv. to  $\neg((\neg A) \vee (\neg B))$  which is equiv. to  $A \wedge B$

So we can get  $\neg$  and  $\wedge$  from  $\downarrow$ . Since  $\{\neg, \wedge\}$  is adequate,  $\{\downarrow\}$  is also adequate.

# Lect. #4 Formal Deductive Systems

A formal deductive system (FDS) is a 4-tuple  $\mathcal{S} = \langle \Sigma, \Phi, P, R \rangle$  where

1.  $\Sigma$  is a denumerable set of symbols called the alphabet
2.  $\Phi$  is a set of strings of symbols in  $\Sigma$  of a special form. The strings in  $\Phi$  are called formulas
3.  $P$  is a special set of formulas which are called axioms
4.  $R$  is a finite set of relations on  $\Phi$  called rules of inferences.

Def. A deduction (or formal proof) in  $\mathcal{S}$  is a finite sequence  $A_1, A_2, \dots, A_n$  of formulas such that for each  $i$ ,  $A_i$  is an axiom or  $A_i$  follows from some of the preceding formulas by a rule of inference

Def. A formula  $A$  is said to be deducible (or provable) in the system  $\mathcal{S}$  if there is a deduction  $A_1, \dots, A_n$  in  $\mathcal{S}$  with  $A_n = A$ .

Def. A formula  $A$  is said to be deducible in  $\mathcal{S}$  from a set of formulas  $H$ , if  $A$  is deducible in  $\mathcal{S}_H = \langle \Sigma, \Phi, P \cup H, R \rangle$

The formulas in  $\Gamma$  are called hypotheses (15)

$\vdash_{\mathcal{L}} A$  means "A is provable in  $\mathcal{L}$ "  
 $\Gamma \vdash_{\mathcal{L}} A$  means "A is provable in  $\mathcal{L}$  from  $\Gamma$ "  
Sometimes we will drop the " $\mathcal{L}$ " and just write " $\vdash$ "

### Some properties of provability

1. If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash A$ , then  $\Delta \vdash A$
2.  $\Gamma \vdash A$  if and only if there is a finite subset  $\Gamma_F$  of  $\Gamma$  such that  $\Gamma_F \vdash A$
3. If  $\Delta \vdash A$  and for each  $B$  in  $\Delta$ ,  $\Gamma \vdash B$ , then  $\Gamma \vdash A$ .

### A F.D.S. for Prop. Logic

$$L = \langle \Sigma, \Phi, P, R \rangle$$

$\Sigma$  : - statement letters :  $A_1, A_2, A_3, \dots$   
primitive connectives :  $\neg, \Rightarrow$   
parentheses :  $(, )$

$\Phi$  : The formulas are defined recursively as follows

- (a) All stat. letters are formulas
- (b) If  $A$  and  $B$  are formulas, then so are  $(\neg A)$  and  $(A \Rightarrow B)$ .

$P$  : If  $A, B, C$  are any formulas then all of the following are axioms:

- (A1) :  $(A \Rightarrow (B \Rightarrow A))$
- (A2) :  $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$
- (A3) :  $((\neg B) \Rightarrow \neg A) \Rightarrow ((\neg B) \Rightarrow A) \Rightarrow B$

R: The only rule of inference is the Rule of detachment (or Modus Ponens)  
 "From  $A$  and  $A \Rightarrow B$ , infer  $B$ ."

We can introduce the other connectives by definition as follows

- (D1) Use  $A \wedge B$  for  $\neg(A \Rightarrow \neg B)$
- (D2) Use  $A \vee B$  for  $(\neg A) \Rightarrow B$
- (D3) Use  $A \Leftrightarrow B$  for  $(A \Rightarrow B) \wedge (B \Rightarrow A)$

Example of a deduction :

For any formula  $A$  in  $L$ ,  $(A \Rightarrow A)$  is provable in  $L$  (i.e.,  $\vdash_L A \Rightarrow A$ ).

1.  $(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))$  Ax. 2  
 $\underbrace{\quad}_B \quad \underbrace{\quad}_C \quad \underbrace{\quad}_B \quad \underbrace{\quad}_C$
2.  $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$  Ax. 1  
 $\underbrace{\quad}_B$
3.  $(A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)$  from 1. & 2 by MP
4.  $A \Rightarrow (A \Rightarrow A)$  Ax. 1
5.  $(A \Rightarrow A)$  from 3. & 4 by MP

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If we want to prove  $A \Rightarrow B$  in some particular area of mathematics we usually assume  $A$  and then deduce  $B$ . Let's say that  $S$  is the F.D.S. for that area of math. What we are doing is to show that  $B$  is provable in  $S \cup \{A\}$ , and then conclude that  $A \Rightarrow B$  is provable in  $S$ .

Is this a logically valid argument? Yes.

### The Deduction Theorem (Herbrand 1930)

If  $\Gamma, A \vdash_L B$  then  $\Gamma \vdash_L A \Rightarrow B$   
In particular if  $A \vdash_L B$ , then  $\vdash_L A \Rightarrow B$ .

Proof: See textbook p. 30.

A Formal Deductive System is said to be sound if all the provable formulas are true.

### Theorem 4 (Soundness theorem)

If  $A$  is provable in  $L$ , then  $A$  is logically valid (i.e., if  $\vdash_L A$  then  $\models A$ )

Proof: First of all the axioms of  $L$  are logically valid (check by using truth tables). Also MP leads us from logically valid formulas to logically valid formulas. Hence every formula which is provable in  $L$  is logically valid.

Lect. #5 The Deduction Theorem

Let  $\Gamma$  be a set of stat. forms and  $A$  &  $B$  be statement forms. Then  $\Gamma, A \vdash B$  implies  $\Gamma \vdash A \Rightarrow B$

Lemma 6

For any statement forms  $A$  &  $B$ , the stat-form  $(A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)$  is provable in  $L$

Proof See Lemma 1.10 (g), p. 38.

Lemma 7: Let  $A$  be a stat. form and  $B_1, \dots, B_n$  be the stat. letters in  $A$ . For a given assignment of truth values to  $B_1, \dots, B_n$ ; let

$B'_i$  be  $B_i$  if Truth val.  $(B_i) = T$   
and  $B'_i$  be  $\neg B_i$  if Truth val.  $(B_i) = F$

Also let

$A'$  be  $A$  if Truth val.  $(A) = T$   
and  $A'$  be  $\neg A$  if Truth val.  $(A) = F$

Then

$B'_1, B'_2, \dots, B'_n \vdash A'$

Proof: See Lemma 1.12, p. ~~33~~<sup>34</sup>

Ex. Let  $A$  be  $(\neg B \Rightarrow C)$ . Then

$B$	$C$	$\neg B \Rightarrow C$	implies	$B, C$	$\vdash A$
T	T	T		$\neg B, \neg C$	$\vdash A$
F	T	T		$B, \neg C$	$\vdash A$
T	F	T		$\neg B, \neg C$	$\vdash \neg A$
F	F	F		$\neg B, \neg C$	$\vdash \neg A$

A formal deductive system  $S$  is said to be adequate if all the true formulas of  $L$  are provable in  $S$ .

Theorem 5 (Adequateness theorem) (Frege, 1891.)

If the formula  $A$  of  $L$  is logically valid, then  $A$  is provable in  $L$ . (i.e., if  $\models A$ , then  $\vdash_L A$ )

Proof: (Kalmar, 1935) : Suppose  $A$  is logically valid. Let  $B_1, \dots, B_k$  be the stat. letters in  $A$ . Also for any truth values of  $B_1, \dots, B_k$  let  $B'_i$  be  $B_i$  if  $B_i$  has truth value  $T$   
 $B'_i$  be  $\neg B_i$  if " " "  $F$

Then it can be shown that  $B'_1, \dots, B'_k \vdash A$  (see Lemma 1.12). ( $A$  is a taut., so  $A'$  is always  $A$ )

So  $B'_1, \dots, B'_{k-1}, B_k \vdash A$  by giving  $B_k$  truth val.  $T$   
and  $B'_1, \dots, B'_{k-1}, \neg B_k \vdash A$  " " "  $F$

So  $B'_1, \dots, B'_{k-1} \vdash B_k \Rightarrow A$  &  $B'_1, \dots, B'_{k-1} \vdash \neg B_k \Rightarrow A$  by the Deduction theorem. But from Lemma 6

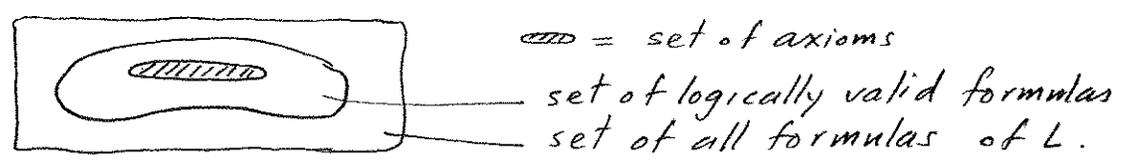
$$\vdash (B_k \Rightarrow A) \Rightarrow ((\neg B_k \Rightarrow A) \Rightarrow A)$$

So by MP we get

$$B'_1, \dots, B'_{k-1} \vdash (\neg B_k \Rightarrow A) \Rightarrow A$$

and by MP again we get  $B'_1, \dots, B'_{k-1} \vdash A$ .

Similarly we can eliminate  $B'_{k-1}$ , etc. all the way down to  $B'_1$ . We will then get  $\vdash A$ .



Def. An F.D.S.,  $S$  which contains " $\neg$ " is said to be consistent if there is no formula  $A$  such that both  $A$  and  $\neg A$  are provable in  $S$

Theorem 8: The F.D.S.  $L$  is consistent (i.e., there is no stat. form  $A$  such that both  $\vdash_L A$  and  $\vdash_L \neg A$ )

Proof: By the Soundness theorem, if  $A$  is provable in  $L$ , then  $A$  is a tautology. But if  $A$  is a tautology,  $\neg A$  will not be a tautology. So by the Completeness theorem  $\neg A$  is not provable in  $L$ . So we can't have  $\vdash_L A$  and  $\vdash_L \neg A$ . Hence  $L$  is consistent.

Def. An F.D.S.,  $S$  is said to be reasonable if for any formulas  $A$  and  $B$ ,  $\neg A \Rightarrow (A \Rightarrow B)$  is provable in  $S$  and  $S$  has MP as a rule of inf.

Ex.  $L$  is a reasonable F.D.S. see Lemma 1.10(c), p. 3

Theorem 9: If  $S$  is a reasonable F.D.S. and  $S$  is inconsistent, then all formulas are provable in  $S$

Proof: Supp.  $S$  is inconsistent, Then for some formula  $\vdash_S A$  and  $\vdash_S \neg A$ . Now let  $B$  be any formula of  $S$ . Then:  $\vdash_S \neg A \Rightarrow (A \Rightarrow B)$  because  $S$  is reasonable. So by MP  $\vdash_S (A \Rightarrow B)$  bec.  $\vdash_S \neg A$ . MP again  $\vdash_S B$  bec.  $\vdash_S A$ . Thus any formula is provable in  $S$ .

## Other Axiomatizations of Prop. Logic :

① Hilbert - Ackermann (1950) :  $L_1 = MP + 4 \text{ Ax. Schemas}$

1.  $(A \vee A) \Rightarrow A$
2.  $A \Rightarrow (A \vee B)$
3.  $(A \vee B) \Rightarrow (B \vee A)$
4.  $(B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow (A \vee C))$

② Rosser (1953) :  $L_2 = MP + 3 \text{ Ax. Sch.}$

1.  $A \Rightarrow (A \wedge A)$
2.  $(A \wedge B) \Rightarrow A$
3.  $(A \Rightarrow B) \Rightarrow (\neg(B \wedge \neg A))$

③  $L_3 = MP + SUB + 3 \text{ Axioms.}$

1.  $A \Rightarrow (B \Rightarrow A)$
2.  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
3.  $(\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)$

④ Kleene (1952) :  $L_4 = MP + 9 \text{ Ax. Sch.} + (\neg\neg A) \Rightarrow A$

Kolmogorov (1930) : I.L. =  $MP + 9 \text{ Ax. Sch.} + \neg A \Rightarrow (A \Rightarrow B)$

## Summary of Ch. I

1. Syntax of Prop. Logic.
2. Semantics of Prop. Logic
3. Algorithmic methods for answering questions about Prop. Logic
4. A Formal Ded. Sys. for Prop. Logic :  $L_1$
5. Properties of the F.D.S.  $L_1$   
Soundedness, Completeness, Consistency
6. Other axiomatizations.