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MHF 4102 - Axiomatic Set Theory — Florida Internat. Univ.

TEST #2 - Spring 2004 — Time: 75 min.

Answer all 6 questions. Justify all of your answers.

(20) 1(a) Define ordinal exponentiation by transfinite recursion.

(b) Prove that for any ordinals α, β, γ $\quad \alpha^{\beta+\gamma} = \alpha^{\beta}\alpha^{\gamma}$.

(You may use any results needed about ordinal add. & mult.)

(17) 2. Simplify the following ordinal expressions as far as possible

(a) $3 + (\omega.2) + \omega^2$ (b) $(\omega+3).(\omega+2)$

(15) 3. (a) Define what are the following cardinal operations

$\kappa + \mu$, $\kappa . \mu$ and κ^{μ}.

(b) Prove that $\kappa + \kappa = \kappa$ if κ is an infinite cardinal.

(15) 4. (a) Write down what the Cantor-Bernstein Theorem says and define what $A \preccurlyeq B$ means.

(b) Let $Seq_F(\mathbb{N})$ = the set of all finite sequences of elements of $\mathbb{N}$. Prove that $Seq_F(\mathbb{N}) \approx \mathbb{N}$.

(15) 5. (a) Write down what the Axiom of Choice (AC) says & what is the Well-ordering Principle (W.O.P).

(b) Prove that W.O.P. $\Rightarrow$ AC.

(18) 6. (a) Let λ be a limit ordinal. Define what is $cof(\lambda)$. Also define what is a strongly inaccessible cardinal.

(b) Prove that if λ is a limit ordinal then $cof(\lambda)$ is always a cardinal.

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Solutions to TEST # 2 — Spring 2004

1 (a) We define α^β by transfinite recursion as follows
(i) $\alpha^0 = 1$, (ii) $\alpha^{\beta+1} = \alpha^\beta . \alpha$, and
(iii) $\alpha^\lambda = \sup\{\alpha^\beta : \beta < \lambda\}$ for limit ordinals $\lambda > 0$.

(b) We will prove $\alpha^{\beta+\gamma} = \alpha^\beta . \alpha^\gamma$ by transfinite induction on γ. Here α and β are fixed but arbitrary. For $\gamma = 0$ we have, $\alpha^{\beta+0} = \alpha^\beta = \alpha^\beta . 1 = \alpha^\beta . \alpha^0$

So the result is true for $\gamma = 0$.

Now suppose the result is true for γ. We will prove it for $\gamma+1$. We have

$$
\begin{aligned}
\alpha^{\beta+(\gamma+1)} &= \alpha^{(\beta+\gamma)+1} && \text{bec. add. is assoc.}\\
&= \alpha^{\beta+\gamma} . \alpha && \text{by (ii) above}\\
&= (\alpha^\beta . \alpha^\gamma) . \alpha && \text{bec. result is true for } \gamma\\
&= \alpha^\beta (\alpha^\gamma . \alpha) && \text{bec. mult. is assoc.}\\
&= \alpha^\beta . \alpha^{\gamma+1} && \text{by (ii) above again.}
\end{aligned}
$$

So if the result is true for γ then it will be true for $\gamma+1$

Finally suppose the result is true for all $\gamma < \lambda$ where λ is a limit ordinal > 0. Then.

$$
\begin{aligned}
\alpha^{\beta+\lambda} &= \sup\{\alpha^{\beta+\gamma} : \gamma < \lambda\} && \text{bec. } \beta+\lambda \text{ is lim. ord.}\\
&= \sup\{\alpha^\beta . \alpha^\gamma : \gamma < \lambda\} && \text{bec. result is true } (\forall \gamma < \lambda)\\
&= \alpha^\beta . \sup\{\alpha^\gamma : \gamma < \lambda\} && \text{bec. } \sup\{\alpha^\gamma : \gamma < \lambda\} \text{ is lim. ord.}\\
&= \alpha^\beta . \alpha^\lambda && \text{bec. } \alpha^\lambda = \sup\{\alpha^\gamma : \gamma < \lambda\}
\end{aligned}
$$

So if the result is true for all $\gamma < \lambda$, then it will be true for λ. Hence by the principle of Transfinite Induction, it is true for all γ. Since α and β were arbitrary it will be true for all α, β and γ.

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2(a) $3 + \omega.2 + \omega^2 = ((3+\omega)+\omega) + \omega^2$
$= (\sup\{3+n : n<\omega\} + \omega) + \omega^2$
$= (\omega+\omega) + \omega^2 = \omega.2 + \omega.\omega$
$= \omega.(2+\omega) = \omega.\sup\{2+k : k<\omega\}$
$= \omega.\omega = \omega^2$

(b) $(\omega+3).(\omega+2) = (\omega+3).\omega + (\omega+3).2$ (left distr. law)
$= \sup\{(\omega+3).n : n<\omega\} + (\omega+3)+(\omega+3)$
$= \sup\{(\omega+3)+(\omega+3)+\cdots+(\omega+3)\ (n\text{ times}) : n<\omega\} + \omega+(3+\omega)+3$
$= \sup\{\omega.n+3 : n<\omega\} + \omega+\omega+3$
$= \omega.\omega + \omega.2 + 3 = \omega^2 + \omega.2 + 3$
because $\omega^2 \ \sup\{\omega.n : n<\omega\} \leq \sup\{\omega.n+3 : n<\omega\}$
$\leq \sup\{\omega.(n+1) : n<\omega\} = \omega^2$

3(a) $\kappa + \mu = |(\kappa\times\{0\}) \cup (\mu\times 1)|$, $\kappa.\mu = |\kappa\times\mu|$
$\kappa^\mu = |\mathcal{F}(\mu,\kappa)| = |$set of all functions from μ to $\kappa|$

(b) We say that an ordinal $\alpha\in\kappa$ is odd if it can be expressed in the form $\lambda+n$ and n is odd, where λ is a limit ordinal. We say α is even if n is even.
Let κ_{odd} = set of all odd ordinals in κ and
κ_{even} = set of all even ordinals in κ.
Define $f : (\kappa\times\{0\})\cup(\kappa\times\{1\}) \to \kappa$ by
$f(\langle\alpha,0\rangle) = \lambda+2n$ if $\alpha = \lambda+n$
$f(\langle\alpha,1\rangle) = \lambda+2n+1$ if $\alpha = \lambda+n$.
Then f is a bijection, provided κ is infinite.
So $\kappa+\kappa = |(\kappa\times\{0\})\cup(\kappa\times\{1\})|$
$= |\kappa|$ because $\kappa \approx (\kappa\times\{0\})\cup(\kappa\times\{1\})$
$= \kappa$
Hence $\kappa+\kappa=\kappa$ for all infinite cardinals κ.

4(a) Cantor-Bernstein Theorem: If $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \approx B$. Here $A \preccurlyeq B$ means that there is an injection from A to B

(b) Define $f: Seq_F(\mathbb{N}) \to \mathbb{N}$ by
$f(\langle a_1, \cdots, a_n \rangle) = 2^n \cdot p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$
where $p_1, p_2, \cdots, p_n, \cdots$ is the sequence of odd prime numbers in increasing order. Note when $n = 0$, $\langle a_1, \cdots, a_n \rangle = \langle \rangle$ = the empty sequence, which has length 0. Now it follows from the Fundamental theorem of Arithmetic that f is an injection. So $Seq_F(\mathbb{N}) \preccurlyeq \mathbb{N}$. Also if we define $g: \mathbb{N} \to Seq_F(\mathbb{N})$ by $g(k) = \langle k \rangle$ then we see that g is also an injection. So $\mathbb{N} \preccurlyeq Seq_F(\mathbb{N})$. So it follows from the Cantor-Bernstein Theorem that $Seq_F(\mathbb{N}) \approx \mathbb{N}$.

5(a) Axiom of Choice (AC): If $\mathcal{A}$ is a set of disjoint non-empty sets then there is a set M which consists of one element of each member of $\mathcal{A}$.
Well-Ordering Principle (WOP): For each set A, we can a relation $\preccurlyeq$ on A such that $\langle A, \preccurlyeq \rangle$ is a well-ordered set.

(b) WOP $\Rightarrow$ AC: Suppose $\mathcal{A}$ is a set of disjoint non-empty sets. Then by WOP we can find a well-ordering $\preccurlyeq$ on $\cup\mathcal{A}$ because $\cup\mathcal{A}$ is a set. Now let each $A \in \mathcal{A}$ let $f(A)$ = smallest element of A according to "$\preccurlyeq$" and let's put $M = \{f(A) : A \in \mathcal{A}\}$. Then M is a set because of the replacement axiom. Also M consists of one element of each member of $\mathcal{A}$. So WOP $\Rightarrow$ AC.

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6. (a) $\operatorname{cof}(\lambda)$ = smallest ordinal θ such that there is a sequence $\langle \alpha_\beta : \beta < \theta \rangle$ of ordinals in λ with $\sup\{\alpha_\beta : \beta < \theta\} = \lambda$

A cardinal κ is strongly inaccessible if
(i) $\kappa > \aleph_0$, (ii) κ is regular, and
(iii) for each $\mu < \kappa$, $2^\mu < \kappa$.

(b) Let $\theta = \operatorname{cof}(\lambda)$. Suppose θ is not a cardinal. Then $|\theta| < \theta$. Now by the definition of $\operatorname{cof}(\lambda)$ we can find a sequence $\langle \alpha_\beta : \beta \in \theta \rangle$ of ordinals in λ such that $\sup\{\alpha_\beta : \beta < \theta\} = \lambda$. Let $\kappa = |\theta|$ and $f : \kappa \to \theta$ be a bijection. Then $\langle \alpha_{f(\gamma)} : \gamma < \kappa \rangle$ will be a sequence of ordinals in λ with $\sup\{\alpha_{f(\gamma)} : \gamma < \kappa\} = \lambda$. But this implies that $\operatorname{cof}(\lambda) \leq \kappa < \theta$ — a contradiction. Hence θ must be a cardinal.