A Vexatious Verification

Section 1.2

2. (b) Show that $5x^2y^2 - 2x^3y^2 = 1$ is an implicit solution to the differential equation

(2.1) $$x \frac{dy}{dx} + y = x^3y^3$$

on the interval $0 < x < 5/2$.

First observe that by doing a little algebra in your head or on your scratch pad, you can easily see that the equation $5x^2y^2 - 2x^3y^2 = 1$ is equivalent to

(2.2) $$y^2 = -1/[2x^2(x - (5/2))].$$

Since the right side of (2.2) is positive when $0 < x < 5/2$, you can obtain two obvious explicit functions defined on the interval $(0,5/2)$. This shows that y in the equation $5x^2y^2 - 2x^3y^2 = 1$ is implicitly a function of x provided when x is restricted to the interval $(0,5/2)$. If you wish, you can verify the two obvious functions are solutions to (2.1) by direct substitution.

Instead of following that course of action, I am going to show you two slightly different verifications by means of implicit differentiation followed by magical algebra. [Implicit solutions usually require this sort of verification.]

The first is brief and brutal, since it depends on the 20-20 hindsight that (2.1) is one of those Bernoulli varmints. Note that (2.1) is equivalent to the ODE

(2.3) $$x \frac{dy}{dx} y^{-3} + y^{-2} = x^3,$$

and $5x^2y^2 - 2x^3y^2 = 1$ is equivalent to the equation

(2.4) $$y^{-2} = 5x^2 - 2x^3.$$

If we differentiate both sides of (2.4) after pretending y is a function of x, we obtain $-2y^{-3}y' = 10x - 6x^2$. By doing the obvious algebra using this last equation, it follows that we have

(2.5) $$xy^{-3}y' = 3x^3 - 5x^2.$$

Finally, (2.4) and (2.5) imply (2.3) is valid --- it all adds up.

The second verification using implicit differentiation is somewhat more pedestrian. Here we first convert the ODE (2.1) above to an equivalent differential form equation doing algebra that is routine. Having done so, we obtain the following:

(2.6) $$(y - x^3y^3)dx + (x)dy = 0.$$

Observe that $5x^2y^2 - 2x^3y^2 = 1$ is equivalent to the equation $(5x^2 - 2x^3)y^2 = 1$. If we now differentiate $(5x^2 - 2x^3)y^2 = 1$ implicitly while pretending that y is a function of x, we produce

$$(10x - 6x^2)y^2 + 2(5x^2 - 2x^3)y(dy/dx) = 0.$$

Taking this and multiplying by dx and dividing by 2y leads to

$$[(5x - 3x^2)y]dx + [5x^2 - 2x^3]dy = 0.$$

Now we shall use the equation $(5x^2 - 2x^3)y^2 = 1$ twice more!
We shall replace both bold expressions below:

$$[(\mathbf{5x - 2x^2} - x^2)y]dx + [\mathbf{5x^2 - 2x^3}]dy = 0.$$

This yields

$$[(x^{-1}y^{-2} - x^2)y]dx + [y^{-2}]dy = 0.$$

At this point, multiplying both sides of the equation immediately above by xy^2 and cleaning things up produces (2.6).//