Directions: Find the general solution of each of the differential equations in Exercises 1 - 22. In each case assume $x > 0$.

19. $x^2y'' + xy' + y = 4\sin\ln x$

Of course to clarify matters, this is really

$$x^2y'' + xy' + y = 4\sin(\ln(x))$$

Why is this silly Euler-Cauchy equation of interest?? The answer in the back of the text,

$$y = c_1\sin(\ln(x)) + c_2\cos(\ln(x)) + \sin(\ln(x))\int\frac{\cos(\ln(x))}{1+x}dx - \cos(\ln(x))\int\frac{\sin(\ln(x))}{1+x}dx$$

is actually quite wide of the mark. Want to play a little Jeopardy?? When you get bored, determine the second order Euler-Cauchy equation for which this is the general solution. Note: If you understand solution structure, you can simplify your task considerably.

In the meantime, let's obtain the solution to Problem 19.

First we reduce the Euler-Cauchy varmint to a constant coefficient animal. Let $x = e^t$ so that $t = \ln(x)$ for $x > 0$, and so as not to overload the symbol y, set $w(t) = y(e^t)$. Thus, $y(x) = w(\ln(x))$ for $x > 0$. After making the substitution and clearing the algebraic dust, we end up with

(*) $$w''(t) + w(t) = 4\sin(t)$$

The solution of this ODE is a routine application of linear techniques, augmented with undetermined coefficient methods.

As usual, we begin by considering the corresponding homogeneous equation:

$$w''(t) + w(t) = 0\ .$$

Since this is a constant coefficient equation, obtaining a fundamental set of solutions to this reduces to obtaining the roots of the auxiliary equation:

$$m^2 + 1 = 0\ .$$

Obviously, we see that the auxiliary polynomial factors as follows:

$$m^2 + 1 = (m + i)(m - i),\ where\ i^2 = -1\ .$$

This means that a fundamental set of solutions to the corresponding homogeneous equations is given by

$$\{\cos(t)\ ,\ \sin(t)\}\ ,$$

and that the complementary solution is

$$w_C(t) = c_1\cos(t) + c_2\sin(t)\ .$$

From the UC theory, since $4\sin(t)$ is part of the complementary solution, we should expect a particular integral to be of the form

$$w_P(t) = At\cos(t) + Bt\sin(t)\ ,$$

where A and B are constants that we will determine by substitution into the (*).

Now, since

$$w_P'(t) = A[\cos(t) - t\sin(t)] + B[\sin(t) + t\cos(t)]$$

and

$$w_P''(t) = A[(-2\sin(t)) - t\cos(t)] + B[2\cos(t) - t\sin(t)] ,$$

by substituting w_P into (*) and simplifying algebraically, we obtain

$$-2A\sin(t) + 2B\cos(t) = 4\sin(t), \textit{for each } t \in \mathbb{R} .$$

From the linear independence of sine and cosine, it follows that $A = -2$ and $B = 0$. Thus,

$$w_P(t) = -2t\cos(t) .$$

With w_P in hand, we may write the general solution to the transformed equation (*):

$$w(t) = w_C(t) + w_P(T) = c_1\cos(t) + c_2\sin(t) - 2t\cos(t)$$

All we need do now is transform back to x's and y's.

With all parts of the puzzle in hand, we may write the general solution to #19:

$$y(x) = w(\ln(x)) = c_1\cos(\ln(x)) + c_2\sin(\ln(x)) - 2\ln(x)\cos(\ln(x))$$

for $x > 0$.

Obviously this is not the only route to this solution. Variation of parameters may also be utilized.

Jeopardy Answer:

$$x^2y'' + xy' + y = \frac{x}{x+1}$$