Floyd's DE Notes: Some More Linear Second Order Stuff

The Homogeneous Constant Coefficient Case

Recall that the key to solving the linear second order equation

$$y'' + a_1(x)\cdot y' + a_2(x)\cdot y = F(x),$$

is to obtain a fundamental set of solutions to the corresponding homogeneous linear equation

$$y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0.$$

We shall now deal with a case where the homogeneous equation is very easy to solve, the case where all the coefficient functions are constant functions. In this case, we may suppose that the homogeneous equation looks like this:

$$(*) \qquad a\cdot y'' + b\cdot y' + c\cdot y = 0,$$

where a,b, and c are real numbers with $a \neq 0$. Inspired by the first order version of the problem,

$$a\cdot y' + b\cdot y = 0,$$

which you can easily solve using the material on linear equations from Chapter 2, and which has a general solution consisting of

$$y = C\cdot e^{-(b/a)\cdot x},$$

you might reasonably guess that (*) has at least one solution which looks like

$$y = e^{m\cdot x}.$$

It turns out that if we substitute this guess into equation (*) above, we obtain

$$am^2\cdot e^{m\cdot x} + bm\cdot e^{m\cdot x} + c\cdot e^{m\cdot x} = 0,$$

or

$$(am^2 + bm + c)\cdot e^{m\cdot x} = 0.$$

This last equation implies that if our guess is a solution to (*), then m must be a solution to the quadratic equation

$$(**) \qquad (am^2 + bm + c) = 0,$$

an equation that is called the *auxiliary or characteristic equation of the o.d.e. (*)*.

Since the earlier theory indicates that any fundamental set of solutions to (*) should have two members, the solutions to (**) may very well provide us with the key to a general solution to (*).

Consequently, it is well worth our while to look at the solutions to (**) in a systematic way.

Recall from your earlier algebra courses that the nature of the solutions to the quadratic equation (**) depends on the sign of the discriminant,

$$D = b^2 - 4 \cdot a \cdot c.$$

You have two distinct real roots if D is positive, one real root of multiplicity 2 when D is zero, and two complex roots that are conjugates of each other if D is negative. Now it turns out that each of these possibilities leads to a distinctly different situation in obtaining a fundamental set of solutions to (*). Consequently, we shall deal with each case in turn.

Case 1: $b^2 - 4 \cdot a \cdot c > 0$

If $b^2 - 4 \cdot a \cdot c > 0$, then the auxiliary equation (**) has two distinct real roots, say r and s. It turns out that it is very easy to see that the two functions $y_1 = e^{r \cdot x}$ and $y_2 = e^{s \cdot x}$ are linearly independent solutions to (*). [That they are solutions is a consequence of r and s being roots of the auxiliary equation. Since the Wronskian of the two functions is $W(y_1, y_2) = (s - r)e^{(r+s) \cdot x}$ and $s - r \neq 0$, it follows that y_1 and y_2 are linearly independent. What we may now conclude is this: $\{ e^{r \cdot x}, e^{s \cdot x} \}$ is a fundamental set of solutions to (*) in this case, and the general solution is given by $y(x) = c_1 \cdot e^{r \cdot x} + c_2 \cdot e^{s \cdot x}$.

Case 2: $b^2 - 4 \cdot a \cdot c = 0$

If $b^2 - 4 \cdot a \cdot c = 0$, then the auxiliary equation (**) has one real root, $r = -(b/2a)$, of multiplicity 2. This means, of course, that the left side of the auxiliary equation actually factors so that (**) is equivalent to $a(m - (-b/2a))^2 = 0$. Unfortunately, now the auxiliary equation only gives us one solution, namely, $y_1 = e^{r \cdot x}$. We need two *linearly independent* solutions to build a fundamental set of solutions. To get a second, linearly independent solution we must use the magic encapsulated in the technique of

Reduction of Order:

A Useful Technique to get 'Missing' Solutions.

If f(x) is a non-trivial solution to the homogeneous equation (*) above, the substitution $y(x) = v(x) \cdot f(x)$ allows us to reduce (*) to a first order equation in v'. Set $w = v'$ to get a first order linear homogeneous equation. [Here, 'non-trivial' means 'not the zero function'.]

You'll recall that Georges's Notes gave you no examples illustrating this. Now we shall see reduction of order in use, perhaps somewhat abstractly, but still, in use.

Here we go. Set $y = v \cdot e^{r \cdot x}$. Then, substituting y into (*), we obtain by doing the required differentiations that

$$0 = a \cdot y'' + b \cdot y' + c \cdot y$$

is equivalent to

$$0 = a \cdot (v'' + 2rv' + r^2 v) \cdot e^{r \cdot x} + b \cdot (v' + vr) \cdot e^{r \cdot x} + c \cdot v \cdot e^{r \cdot x}.$$

$$= (ar^2 + br + c) \cdot v \cdot e^{r \cdot x} + (av'' + (2ar + b)v') \cdot e^{r \cdot x}.$$

In the last equation above, since r is a root of (**), we have $ar^2 + br + c = 0$, and because we actually have $r = -(b/2a)$, the coefficient on v' in the second term above is also zero. Consequently, the mess above reduces down to this:

$$0 = av'' \cdot e^{r \cdot x},$$

or

$$0 = v'',$$

since $a \neq 0$.

This last second order differential in v can be transformed into a first order linear homogeneous equation by means of a simple substitution. Let $w = v'$. Then by substituting w into the differential equation $0 = v''$, we obtain $w' = 0$.

[The reduction to first order linear homogeneous is typical. The utter, stark simplicity of **$w' = 0$** is not usual, however. Incidentally, here is a silly question: What is the integrating factor for $w' = 0$???]

We may solve $w' = 0$ by simply integrating to get $w = c$, a constant. Consequently, $v' = c$. Integrating one more time yields $v = cx + d$, where d is a new constant. This means, finally, that we have

$$y = (cx + d) \cdot e^{r \cdot x},$$

or

$$y = c \cdot x \cdot e^{r \cdot x} + d \cdot e^{r \cdot x},$$

a linear combination of the two functions $e^{r \cdot x}$ and $x \cdot e^{r \cdot x}$. Observe that the first function is our original solution and the second is new. By computing the wronskian of these two functions, you can see that they are linearly independent. Consequently, in this case a fundamental set of solutions for (*) is $\{ e^{r \cdot x}, x \cdot e^{r \cdot x} \}$ and the general solution is given by $y(x) = c_1 \cdot x \cdot e^{r \cdot x} + c_2 \cdot e^{r \cdot x}$.

Case 3: $b^2 - 4 \cdot a \cdot c < 0$

If $b^2 - 4 \cdot a \cdot c < 0$, then the auxiliary equation (**) has two complex roots that are conjugates since a, b, and c are real numbers. Without any loss of generality, we may assume the two roots are $r + s \cdot i$ and $r - s \cdot i$, where r and s are a pair of real numbers. As in Case 1, you get two linearly independent solutions, $e^{(r + s \cdot i) \cdot x}$ and $e^{(r - s \cdot i) \cdot x}$. Unfortunately, these are not usable by us because these are, in fact, complex-valued functions. It turns out that Euler's identity, $e^{i \cdot x} = \cos(x) + i \cdot \sin(x)$, bails us out. [**Note**: "Euler" is pronounced "Oiler"!!] This identity connects the complex exponential function with our old friends sine and cosine. As a result, we can also solve the linear system,

$$e^{i \cdot x} = \cos(x) + i \cdot \sin(x)$$

$$e^{-i \cdot x} = \cos(x) - i \cdot \sin(x),$$

created using the oddness of sine and the evenness of cosine, to obtain sin(x) and cos(x). When we do, we get

$$\cos(x) = (1/2) \cdot (e^{i \cdot x} + e^{-i \cdot x})$$

and

$$\sin(x) = (1/2i) \cdot (e^{i \cdot x} - e^{-i \cdot x}).$$

Since sine and cosine are complex linear combinations of $e^{i \cdot x}$ and $e^{-i \cdot x}$ above, it turns out that we can uncover two real-valued functions $y_1 = e^{r \cdot x} \cdot \cos(s \cdot x)$ and $y_2 = e^{r \cdot x} \cdot \sin(s \cdot x)$ that are solutions to (*) in this case, and they are actually linearly independent. Consequently, in this case a fundamental set of real-valued solutions to (*) is $\{ e^{r \cdot x} \cdot \cos(s \cdot x), e^{r \cdot x} \cdot \sin(s \cdot x) \}$, and the general solution is $y(x) = c_1 \cdot e^{r \cdot x} \cdot \cos(s \cdot x) + c_2 \cdot e^{r \cdot x} \cdot \sin(s \cdot x)$.

[**Exercise**: Do the plug and chug to verify that y_1 and y_2 above are linearly independent by computing their wronskian, and substitute each separately into (*) to verify each is a solution. In doing the last part, you will have to use the hypothesis that $r + s \cdot i$ and $r - s \cdot i$ are roots to (**).]

At this point we have finished our guided tour to the solutions to the second order linear homogeneous constant coefficient equation. Here are a couple of things you must keep in mind.

(1): To actually solve a second order constant coefficient equation (*), you must obtain the roots to the auxiliary equation (**) by factoring or using your old friend, the quadratic formula.

(2): To understand the solutions of the general n^{th} order constant coefficient equation, it really helps to read the text carefully. The alternative is to re-invent that wheel yourself by guessing the pattern and doing the required induction argument. That, of course, is a fun thing to do, but given your time constraints, it might be wiser to crack the book.