Some Useful Linear Algebra Concepts

(1) Linear Combination

If $f_1,...,f_n$ are functions and $c_1,...,c_n$ are numbers, the function f defined by

$$f(x) = c_1 \cdot f_1(x) + ... + c_n \cdot f_n(x)$$

is called a **linear combination** of $f_1,...,f_n$.

Example(s):

(a) $f(x) = 3 \cdot \sin(5x) - 8 \cdot \cos(9x) + 32 \cdot e^x$

Here f is a linear combination of the functions sin(5x), cos(9x), and e^x. Of course, f is also a linear combination of the functions $3 \cdot \sin(5x)$, $8 \cdot \cos(9x)$,and $32 \cdot e^x$!!

(b) $g(x) = ax^2 + bx + c$

Your favorite quadratic function is merely a linear combination of the functions x^2, x, and '1'. This is a formal linear combination.

You will see many many more linear combinations of all sorts of functions throughout your study of linear differential equations. Just hold on for the ride. This construct is probably the single most pervasive in dealing with linear DE's. Gradually you'll begin to understand why.

(2) Linear Dependence

Let $f_1,...,f_n$ be n functions. If there are n numbers $c_1,...,c_n$, not all equal to zero, such that

$$c_1 \cdot f_1(x) + ... + c_n \cdot f_n(x) = 0 \text{ for all } x,$$

then $f_1,...,f_n$ are said to be **linearly dependent**.

You can paraphrase this as follows: A collection of n functions $f_1,...,f_n$ is linearly dependent if the zero function can be expressed as a non-trivial linear combination of them. This is of interest because the zero function can always be written as a linear combination of any functions *trivially*, that is with all the numerical coefficients $c_1,...,c_n$ equal to the number 0.

(3) Linear Independence

Let $f_1,...,f_n$ be n functions. If

$$c_1 \cdot f_1(x) + ... + c_n \cdot f_n(x) = 0$$

being true for all x implies that $c_1 = c_2 = ... = c_n = 0$, then we say that $f_1,...,f_n$ are **linearly independent**.

You can paraphrase this as follows: A collection of n functions $f_1,...,f_n$ is linearly independent if the zero function can be expressed only as a trivial linear combination of them. This is of interest because the zero function can always be written as a linear combination of any functions *trivially*, that is with all the numerical coefficients $c_1,...,c_n$ equal to the number 0. When the functions $f_1,...,f_n$ are linearly independent, this is the only way that the zero function can be written as a linear combination using the given functions.

It doesn't take a rocket scientist to see that the two concepts of linear dependence and linear independence are logical negations of each other. What makes these two concepts so very important is that they are the keys to the matter of uniqueness of representation for functions expressed as linear combinations of other functions. Specifically, if the family of functions $f_1,...,f_n$ are linearly independent, then if f is a function so that

$$f(x) = c_1 \cdot f_1(x) + ... + c_n \cdot f_n(x)$$

for some numbers $c_1,...,c_n$, and if we also have

$$f(x) = d_1 \cdot f_1(x) + ... + d_n \cdot f_n(x)$$

for some numbers $d_1,...,d_n$, then we must have $c_1 = d_1,...,c_n = d_n$. In short, we have a very nice sort of uniqueness that will turn out to be exceedingly useful for us, as you will see in due time.

Example(s):

(a) Let $f_1(x) = 3 \cdot \sin(x) - 4 \cdot \cos(x)$, $f_2(x) = \sin(x)$, and $f_3(x) = \cos(x)$. f_1, f_2,and f_3 are linearly dependent because

$$1 \cdot f_1(x) + (-3)f_2(x) + (4)f_3(x) = 0$$

for every real number x.

(b) On the other hand f_2 and f_3 above are linearly independent, since the equation $a \cdot f_2(x) + b \cdot f_3(x) = 0$ being true for every real number x implies that $a = b = 0$. Here's why: If the equation is true for every x, it's true when $x = 0$ and when $x = \pi/2$. This leads to an easy to solve linear system with the unique solution of $a = 0$ and $b = 0$.

A Tool for Dealing with Linear Independence: The Wronskian.

Let f_1 and f_2 be two functions that are differentiable. The **Wronskian of f_1 and f_2** is the function $W(f_1,f_2)$ defined by the equation

$$W(f_1,f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1(x)f_2'(x) - f_2(x)f_1'(x).$$

How one defines the wronskian of n functions in terms of n x n determinants is fairly obvious.

Some Useful Facts Concerning the Wronskian and Solutions of 2^{nd} Order Linear Equations

(1) If y_1 and y_2 are two functions with $W(y_1,y_2) \neq 0$, then y_1 and y_2 are linearly independent.

[In general, the converse is false. Fortunately, we mostly will have to use this for solutions to homogeneous equations.]

(2) If y_1 and y_2 are solutions to the homogeneous equation

(*) $$y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0,$$

then y_1 and y_2 are linearly independent if, and only if

$$W(y_1,y_2) \neq 0.$$

(3) If y_1 and y_2 are solutions to the homogeneous equation and

$$W(y_1,y_2) \neq 0,$$

then every solution to (*) is a linear combination of y_1 and y_2, that is, every solution y can be written in the form $y = c_1y_1 + c_2y_2$ for appropriate constants c_1 and c_2. The set $\{y_1, y_2\}$ is called a **fundamental set of solutions** to (*).

(4) If $\{y_1, y_2\}$ is any fundamental set of solutions to (*) and y_p is any single solution to the differential equation

(**) $$y'' + a_1(x)\cdot y' + a_2(x)\cdot y = F(x),$$

then every solution to (**) is of the form

$$f(x) = y_p(x) + c_1\cdot y_1(x) + c_2\cdot y_2(x).$$

The general solution to the corresponding homogeneous equation (*) is sometimes called the complementary solution and denoted by y_c. In these terms, the general solution to (**) is sometimes written as

$$f(x) = y_p(x) + y_c(x).$$

Reduction of Order: A Useful Technique to get 'Missing' Solutions.

If f(x) is a non-trivial solution to the homogeneous equation (*) above, the substitution $y(x) = v(x)\cdot f(x)$ allows us to reduce (*) to a first order equation in v'. Set $w = v'$ to get a first order linear homogeneous equation. [Here, 'non-trivial' means 'not the zero function'.] You will see an application of this in Floyd's Notes, coming up.