More Linear Second Order Stuff : Variation of Parameters

By contrast with the *Method of Undetermined Coefficients*, which is essentially restricted to constant coefficient equations and a small class of driving functions, the technique of variation of parameters is quite general.

Again, we shall restrict our attention to the second order case. To simplify matters, we shall assume the equation is normalized so that the coefficient function on the second derivative term is "1", the function that is constantly one. Thus, we are considering the equation

$$(*)\qquad y'' + a_1(x)\cdot y' + a_2(x)\cdot y = F(x),$$

where the coefficient functions, a_1 and a_2, and the driving function, F, are continuous.

Suppose now that we have already dealt with the corresponding homogeneous equation,

$$(**)\qquad y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0,$$

and have in hand a fundamental set of solutions to (**), $\{y_1, y_2\}$. Then, inspired by the form of the solution to the first order linear differential equation, $y' + a(x)\cdot y = F(x)$,

$$y(x) = [\mu(x)]^{-1}\int F(x)\mu(x)dx + c[\mu(x)]^{-1},$$

where μ is an integrating factor, we might guess that a particular integral looks like this:

$$(***)\qquad y_p(x) = v_1(x)\cdot y_1(x) + v_2(x)\cdot y_2(x),$$

where v_1 and v_2 are functions whose identity we shall eventually reveal.

The reason for the guess is this. It turns out that the function $[\mu(x)]^{-1} = (1/\mu(x))$ is a non-trivial solution to the corresponding homogeneous equation, $y' + a(x)\cdot y = 0$, and the first summand of y(x) is, in fact, $y_p(x)$. Look at its form. The function y_p consists of a product of a non-trivial solution to the corresponding homogeneous equation and another function. Now mix in "linear". OK??

Let's now attack the problem of identifying v_1 and v_2. To do this, we shall pretend y_p is a solution to (*) and see what that forces on us. Obviously, the first thing we might want to do is compute the first and second derivatives of $y_p = v_1\cdot y_1 + v_2\cdot y_2$.

Clearly, $y_p' = v_1\cdot y_1' + v_1'\cdot y_1 + v_2\cdot y_2' + v_2'\cdot y_2$. Although it is not very intuitive at this stage, it turns out that it is desirable to have $v_1'\cdot y_1 + v_2'\cdot y_2 = 0$. This simplifies the first derivative of y_p in a critical way. A slightly more advanced perspective on this, where (*) is re-written as a first order linear ODE involving vector-valued functions and matrix coefficients, reveals that $v_1'\cdot y_1 + v_2'\cdot y_2 = 0$ is a necessary condition for y_p having the form (***) to be a solution to (*). Because we shall be thinking about this with v_1 and v_2 as unknown functions, we will write this as

$$(****)\qquad y_1\cdot v_1' + y_2\cdot v_2' = 0.$$

Now if we use (****), y_p' simplifies to

$$y_p' = v_1\cdot y_1' + v_2\cdot y_2',$$

and thus,

$$y_p'' = v_1 \cdot y_1'' + v_2 \cdot y_2'' + v_1' \cdot y_1' + v_2' \cdot y_2'.$$

After we substitute y_p into (*) using y_p' and y_p'' as above, use strongly the assumption that y_1 and y_2 are solutions to the corresponding homogeneous equation (**), and perform a little algebraic magic, we obtain a second equation in v_1' and v_2':

(*****) $$y_1' \cdot v_1' + y_2' \cdot v_2' = F(x).$$

It now turns out that v_1' and v_2' have unique solutions in the linear system consisting of equations (****) and (*****). We know that this system has a unique solution because the determinant of the coefficient matrix is the Wronskian, $W(y_1,y_2)$, and $W(y_1,y_2) \neq 0$. So why is $W(y_1,y_2) \neq 0$?? Remember this: we are pretending that $\{y_1, y_2\}$ is a fundamental set of solutions to (**).

Finally, after we solve the linear system consisting of (****) and (*****), we can obtain v_1 and v_2 by doing simple integrations. At the very worst, we might actually have to use the Fundamental Theorem of Calculus.

We'll now look at a simple example. Before we begin, I'll warn you that texts frequently deal with the linear system by using Cramer's Rule, which gives the solution neatly in terms of determinants. Although this gives a neat theoretical solution, my experience shows that you would be far wiser dealing with the linear system using naive Algebra II techniques. It turns out that Cramer's Rule leaves you with horrors to integrate. Doing naive algebra tends to clean up things so that when you are ready to integrate, the integrands are not so intimidating. This is particularly true for the small systems you will be handling here. For larger systems such as found in Signals and Systems type courses, you will learn appropriate linear algebra tools to deal with the increased complexity.

Simple Example

$$y'' + y = \tan(x)$$

Since the corresponding homogeneous equation is $y'' + y = 0$, which has as its auxiliary equation, $m^2 + 1 = 0$, a fundamental set of solutions to the corresponding homogenous equation is $\{ \sin(x), \cos(x) \}$.

Observe that $\tan(x)$ is not an undetermined coefficient function. Using the technique of variation of parameters, we'd expect to find a particular integral of the form

$$y_p(x) = v_1(x) \cdot \sin(x) + v_2(x) \cdot \cos(x).$$

If y_p, as above, is to be a solution to $y'' + y = \tan(x)$, then v_1' and v_2' must satisfy the following system:

$$\sin(x) \cdot v_1' + \cos(x) \cdot v_2' = 0$$

$$\cos(x) \cdot v_1' - \sin(x) \cdot v_2' = \tan(x)$$

Notice, please, that these two equations are simply (****) and (*****) above with $y_1(x) = \sin(x)$ and $y_2(x) = \cos(x)$.

Now we shall solve this system using simple Algebra II techniques. To this end, solve for v_1' in the first equation and substitute the result into the second. You get the linear system

$$v_1' = -(\cos(x)/\sin(x))\cdot v_2'$$

$$-\cos(x)\cdot(\cos(x)/\sin(x))\cdot v_2' - \sin(x)\cdot v_2' = \tan(x).$$

Observe that the second equation just above is now a single variable linear equation. If you multiply it by sin(x) and then apply your favorite Pythagorean identity, $\sin^2(x) + \cos^2(x) = 1$, the second equation magically becomes $-v_2' = \sin(x)\cdot\tan(x)$. Consequently, we get $v_2' = -\sin(x)\cdot\tan(x)$ and $v_1' = \sin(x)$ after the algebraic dust settles. This gives us the kind of linear system we like:

$$v_1' = \sin(x)$$

$$v_2' = -\sin(x)\cdot\tan(x),$$

where the solution is obvious.

To obtain v_1 and v_2, we need only integrate. Thus,

$$v_1 = \int\sin(x)dx = -\cos(x) + c,$$

and

$$v_2 = -\int\sin(x)\tan(x)dx = -\int\sin^2(x)/\cos(x)dx$$

$$= -\int(1 - \cos^2(x))/\cos(x)dx$$

$$= \int\cos(x) - \sec(x)dx$$

$$= \sin(x) - \ln|\tan(x) + \sec(x)| + d.$$

Finally, observe that constants of integration, c and d above, may safely be set equal to zero in order to obtain a simple particular integral for $y'' + y = \tan(x)$. Their contribution to the general solution may be handled by the arbitrary constants that appear as part of y_c. Just do the algebra ... . After the algebraic dust settles, we have

$$y_p(x) = -(\ln|\tan(x) + \sec(x)|)\cdot\cos(x).$$

If you want the general solution, you get something resembling

$$y(x) = c_1\cdot\sin(x) + c_2\cdot\cos(x) - (\ln|\tan(x) + \sec(x)|)\cdot\cos(x).$$

Finally, it is time to turn to our FAQ file.

Question: "Since variation of parameters is more general than the method of undetermined coefficients, do I *really* need to learn that UC function noise??"

Answer: "No, but if you are pressed for time, like at exam time, you might just want to have the method of UC functions in your arsenal. For both you have to solve linear systems, but with the method of UC functions, you don't have to do any integrations. You do have to be able to do differentiations correctly, though."

The noise above is dedicated to the memory of Sub-Tropical Depression #1, which bathed us to the point where we were enisled at home in verdant, soggy Miami Springs, thus making the generation of these notes necessary and possible. Eventually Sub-Tropical Depression #1 moved out to sea, got its act together, and became Tropical Depression Leslie. Oh my, tropical depression. October, 2000.