Directions: Find the general solution of each of the differential equations in Exercises 1 - 34.

23. $y''' - 3y'' + 4y = 4e^{x} - 18e^{-x}$

The solution of this ODE is a routine application of linear techniques, and in particular, undetermined coefficient methods. What I aim to do additionally is to demonstrate how one can use linearity to cut the problem of finding a particular integral for the ODE into smaller, easier to chew, sub-problems.

As usual, we begin by considering the corresponding homogeneous equation:

$$y''' - 3y'' + 4y = 0 \ .$$

Since this is a constant coefficient equation, obtaining a fundamental set of solutions to this reduces to obtaining the roots of the auxiliary equation:

$$m^3 - 3m^2 + 4 = 0 \ .$$

A reasonable way to attack this third degree polynomial is to see if it has any rational roots. [There is a complicated formula that ...] It follows from the Rational Root Theorem, found in Appendix 2 of Ross's text, the possible rational roots are given by p/q, where p is an integer factor of 4 and q is an integer factor of 1. Thus the possibilities are

$$\pm 1,\ \pm 2,\ or\ \pm 4 \ .$$

Starting with the smallest values and working upward, by using either synthetic division or old-fashioned long division, you can easily see that m = -1 is a root of the auxiliary equation and read off the actual quotient. Consequently, we see that the auxiliary polynomial factors as follows:

$$m^3 - 3m^2 + 4 = (m - (-1))(m^2 - 4m + 4) = (m - (-1))(m - 2)^2 \ .$$

This means that a fundamental set of solutions to the corresponding homogeneous equations is given by

$$\{\ e^{-x}\ ,\ e^{2x},\ xe^{2x}\ \}\ ,$$

and that the complementary solution is

$$y_C = c_1e^{-x} + c_2e^{2x} + c_3xe^{2x} \ .$$

From the UC theory, we should expect a particular integral to be of the form

$$y_P = Ae^{x} + Bxe^{-x}\ ,$$

where A and B are constants that we can determine by substitution into the ODE #23, since e^{-x} is part of the complementary solution. Instead of doing that, however, to contain and localize the potential algebraic mess and demonstrate technique, we shall obtain a particular integral as

$$y_P(x) = f_1(x) - f_2(x) ,$$

where f_1 is a particular integral for the linear ODE

(1) $$y''' - 3y'' + 4y = 4e^x$$

and f_2 is a particular integral for the linear ODE

(2) $$y''' - 3y'' + 4y = 18e^{-x} .$$

Be clear about this: The reason we are able to do this is that the left side of all of the ODE's, the linear operator part of the game, is the same for all.

To determine f_1 first, from the UC theory, we shall set

$$f_1(x) = Ae^x$$

where A is a constant that we'll determine. It follows that if f_1 is a particular integral for the linear ODE (1), then

$$\begin{aligned} 4e^x &= 4f_1 - 3f_1'' + f_1''' \\ &= 4Ae^x - 3Ae^x + Ae^x \\ &= 2Ae^x \text{ , for each } x \in \mathbb{R}. \end{aligned}$$

Consequently, $A = 2$ and

$$f_1(x) = 2e^x .$$

Obtaining f_2 explicitly is slightly messier since e^{-x} is part of the complementary solution space. This means that instead of a multiple of e^{-x}, we need to set

$$f_2(x) = Bxe^{-x}$$

where B is a constant that we'll determine. It follows that if f_2 is a particular integral for the linear ODE (2), then

$$\begin{aligned} 18e^{-x} &= 4f_2 - 3f_2'' + f_2''' \\ &= 4Bxe^{-x} - 3[(-2)Be^{-x} + Bxe^{-x}] + [3Be^{-x} - Bxe^{-x}] \\ &= 9Be^{-x} \text{ , for each } x \in \mathbb{R}. \end{aligned}$$

Consequently, $B = 2$ and

$$f_2(x) = 2xe^{-x} .$$

Putting the pieces together, we have

$$y_P(x) = f_1(x) - f_2(x) = 2e^x - 2xe^{-x} .$$

Finally, with all parts of the puzzle in hand, we may write the general solution to #23:

$$y(x) = y_C(x) + y_P(x) = c_1e^{-x} + c_2e^{2x} + c_3xe^{2x} + 2e^x - 2xe^{-x} .$$