Moving to Zero

When dealing with an initial value problem like

(*) $$y''(x) + a_1(x)\cdot y'(x) + a_2(x)\cdot y(x) = F(x), \text{ with}$$

$$y(x_0) = c_0$$

and

$$y'(x_0) = c_1,$$

it is sometimes convenient to transform the problem so that the initial conditions are at the origin of the number line. This is particularly true when finding power series solutions, working with the Frobenius machine, or using Laplace transforms to solve the differential equation or initial value problem.

To transform the problem is very easy. We simply set $x = t + x_0$, and let $w(t) = y(t + x_0)$ to avoid overloading the symbol 'y'. Then $w'(t)= y'(t + x_0)$ and $w''(t)= y''(t + x_0)$. Thus, replacing 'x' in equation (*) with '$t + x_0$', we obtain

$$y''(t + x_0) + a_1(t + x_0)\cdot y'(t + x_0) + a_2(t + x_0)\cdot y(t + x_0) = F(t + x_0),$$

or

$$w''(t) + a_1(t + x_0)\cdot w'(t) + a_2(t + x_0)\cdot w(t) = F(t + x_0).$$

Since $x = t + x_0$ implies that $x = x_0$ if, and only if $t = 0$, the transformed initial value problem is

(**) $$w''(t) + a_1(t + x_0)\cdot w'(t) + a_2(t + x_0)\cdot w(t) = F(t + x_0), \text{ with}$$

$$w(0) = c_0$$

and

$$w'(0) = c_1.$$

Once we obtain the solution to the transformed problem (**), the solution to the original problem (*) is given in terms of 'w' by the equation

$$y(x) = w(x - x_0).$$

Note: *If the equation in (*) is a constant coefficient equation*, only the right side, the driving function, gets changed by the transformation. In this case, the left sides of (*) and (**) will be the same except for being in terms of 'w' and 't' instead of 'y' and 'x'.

Here are a couple of simple examples. The second one has a constant coefficient differential equation.

(1) If

$$y''(x) + 3x^2 \cdot y'(x) + \cos(4x) \cdot y(x) = \ln(x + 5), \text{ with}$$

$$y(3) = -2,$$

and

$$y'(3) = 4,$$

by letting $x = t + 3$ and $w(t) = y(t + 3)$, we obtain the initial value problem

$$w''(t) + 3(t + 3)^2 \cdot w'(t) + \cos(4(t + 3)) \cdot w(t) = \ln(t + 8), \text{ with}$$

$$w(0) = -2,$$

and

$$w'(0) = 4.$$

Once we solve the transformed initial value problem, the solution to the original problem is given by $y(x) = w(x - 3)$.

(2) If

$$y''(x) + 20 \cdot y'(x) - 4 \cdot y(x) = \tan^{-1}(30 \cdot x), \text{ with}$$

$$y(-7) = 23$$

and

$$y'(-7) = -8,$$

by letting $x = t + (-7)$ and $w(t) = y(t - 7)$, we obtain the initial value problem

$$w''(t) + 20 \cdot w'(t) - 4 \cdot w(t) = \tan^{-1}(30 \cdot (t - 7)), \text{ with}$$

$$w(0) = 23,$$

and

$$w'(0) = -8.$$

Note that the coefficient functions on the left sides of the differential equations are the same. This is typical of constant coefficient equations. Like the first example, once we obtain a solution to the transformed initial value problem, the solution to the original problem is given by $y(x) = w(x - (-7))$.