After solving a few elementary linear constant coefficient ordinary differential equations using the Laplace transform, you may very well wonder whether the algebraic price is worth it. After all, many of the ODE's are actually simple homogeneous equations or have undetermined coefficient driving functions. Frequently, these may be solved more readily without the use of the Laplace transform with minimal algebraic mess.

So why should we bother with the Laplace transform in solving constant coefficient ODE's?? There are a number of valid reasons. One of these is that when the driving function is either piece-wise defined or periodic, but not domesticated like sine or cosine, the Laplace transform truly eases the pain of solution. Then the algebraic price of admission is worth it.

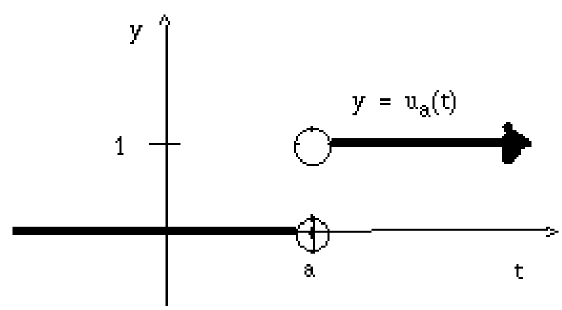
The key to dealing with step functions, and more generally, piece-wise defined functions is a particularly simple step function defined on the real line

$$u(t) = \begin{cases} 0 , & if \quad t < 0 \\ 1 , & if \quad 0 < t , \end{cases}$$

and its translates,

$$u_a(t) = u(t-a) = \begin{cases} 0 , & if \quad t < a \\ 1 , & if \quad a < t , \end{cases}$$

for $a > 0$. You can view a typical one of these functions immediately below.



How these functions are actually defined at the jump point is not horribly important since we intend to use them in computing simple Laplace transforms. We may patch the hole in whatever way is most convenient at the appropriate time.

How do these silly functions assume a central role in computing Laplace transforms of piece-wise defined functions?? There are three parts to a reasonable answer:

(A) The Laplace transform is linear;

(B) it is very easy to write essentially any piece-wise continuous function, $f(t)$, defined on the positive real numbers as a sum of functions of the form

(1) $$g(t-a)\,u_a(t)$$

for suitable choices of nonnegative real numbers a and functions $g(t)$; and

(C) the Laplace transform of each of the products appearing in (1) above is given simply in terms of the Laplace transform of the function $g(t)$. [*Of course, when g(t) is a constant, the transform is merely a multiple of the transform of the unit step.*]

Nearly every elementary differential equations text treats (A) and (C) above adequately, although the definition and notation for the unit step functions may very somewhat from text to text with some merely using explicit translates of the function u above in their formulae. What none do really well is reveal the key to (B). That is what we intend to focus on here.

First, though, to establish notation, let us recall what (A) and (C) typically entail.

(A) *Linearity*: $$\begin{cases} \mathfrak{L}\{g(t)+h(t)\}(s) = \mathfrak{L}\{g(t)\}(s) + \mathfrak{L}\{h(t)\}(t) \\ \text{and} \\ \mathfrak{L}\{c\,g(t)\}(s) = c\,\mathfrak{L}\{g(t)\}(s) \end{cases}$$

for suitable real numbers s whenever g and h are functions with Laplace transforms and c is any number. Moreover, a simple substitution in the defining integral yields

(C) $$\mathfrak{L}\{g(t-a)\,u_a(t)\}(s) = e^{-as}\mathfrak{L}\{g(t)\}(s)$$

for a appropriate real numbers s, provided g is nice enough. One also occasionally encounters the following variant of (C):

(C′) $$\mathfrak{L}\{g(t)\,u_a(t)\}(s) = e^{-as}\mathfrak{L}\{g(t+a)\}(s)\ .$$

[If one does not want to store both in memory, (C) is best, for it is easier to apply when one gets around to dealing with inverse transforms.]

So what about (B)??

Recall, if a function *f* defined on the nonnegative real numbers is piece-wise or sectionally continuous, then for each $R > 0$, there is a partition of the interval $[0,R]$, i.e., a finite set of points from the interval,

$$0 = a_0 < a_1 < a_2 < \dots < a_{n-1} < a_n = R$$

so that *f* is continuous on the interiors of the sub-intervals

$$[a_{k-1}, a_k], \text{ for } k = 1, \dots, n,$$

and has finite one-sided limits at the endpoints. Evidently the sequence of 'jump' points

$$\{a_k\}$$

may be either infinite or finite. For the sake of simplicity initially we shall restrict our attention to the case where this set is finite and we set

$$a_n = \infty.$$

How simple does it get?? If *f* is given by

$$f_k(t) \text{ for } t \in (a_{k-1}, a_k), \text{ for } k = 1, \dots, n,$$

we may write

$$f(t) = \sum_{k=1}^{n} (f_k(t) - f_{k-1}(t))\, u_{a_{k-1}}(t)$$

provided that for convenience, we set

$$f_0(t) = 0.$$

Of course we have ignored the issue of how the function is defined at the finitely many 'bad' points.

Did that go flying over your head?? Let me show you the origin of that *telescoping sum* with *off-on switches* by giving a couple of examples in boring, real-time detail.

First, let's deal with a simple step function. Consider

$$f(t) = \begin{cases} 4, & \text{if } 0 < t < 10 \\ -5, & \text{if } 10 < t < 20 \\ 2\pi, & \text{if } 20 < t < 30 \\ -1, & \text{if } 30 < t. \end{cases}$$

We shall write this function as a linear combination of unit step functions. The process involves writing a sequence of telescoping sums whose summands *switch on* at the appropriate jump points.

To begin, deal with the definition of *f* in the left-most interval. Set

$$f(t) = 4u_0(t).$$

Observe that the equality is only valid for *t*'s satisfying $0 < t < 10$. At this point, we have this:

$$f(t) = \begin{cases} 4, & \text{if } 0 < t < 10 & \leftarrow \textit{OK here}. \\ -5, & \text{if } 10 < t < 20 & \textit{Not OK}: 4 \\ 2\pi, & \text{if } 20 < t < 30 & \textit{Not OK}: 4 \\ -1, & \text{if } 30 < t & \textit{Not OK}: 4. \end{cases}$$

For values of *t* beyond 10 we need to alter the function on the right side of the equal sign. An appropriate thing to do is add a suitable multiple of the unit step function

$$u_{10}(t)$$

which is 0 for each $t < 10$ and 1 for each $t > 10$. Why?? Adding a multiple of this function will not alter what we have already achieved in matching the right side with the left. And what should the multiplier be?? We shall add in the new, desired value, -5, at the same time as we remove the previously existing value, 4. This means that the desired multiplier is (-5 -(4)). At this stage, then, set

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t).$$

For values of $t > 10$ the right side is always -5. We actually are OK in matching the left and right sides for the first two intervals. In fact a bit of thought will reveal to you that we now have this:

$$f(t) = \begin{cases} 4, & \text{if } 0 < t < 10 & \leftarrow \textit{OK here}. \\ -5, & \text{if } 10 < t < 20 & \leftarrow \textit{OK here}. \\ 2\pi, & \text{if } 20 < t < 30 & \textit{Not OK}: -5 \\ -1, & \text{if } 30 < t & \textit{Not OK}: -5. \end{cases}$$

For *t*'s larger than 20, though, we have to make additional adjustments. And NO, don't do the obvious arithmetic yet!! There is a pattern here that we will repeat, and by not doing the arithmetic you might hope to see it.

Since we do not want to mess up what we have done for any *t* with $t < 20$, but do want to make an adjustment for $t > 20$, the tool to use is the addition of a multiple of the unit step function

$$u_{20}(t)$$

As in the previous adjustment, adding a multiple of this function will not alter what we have already achieved in matching the right side with the left. And, as previously, the multiplier should be such that we add in the new, desired value, 2π, at the same time as we remove the previously existing value, -5. This means now that the desired multiplier is (2π - (-5)). At this stage, then, set

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t).$$

Now for values of *t* > 20 the right side is always 2π. Furthermore, we are OK in matching the left and right sides for the first three intervals. Again, a bit of thought will reveal to you that we now have this:

$$f(t) = \begin{cases} 4 \ , \ if \ \ 0 < t < 10 & \leftarrow \textit{OK here.} \\ -5 \ , \ if \ \ 10 < t < 20 & \leftarrow \textit{OK here.} \\ 2\pi \ , \ if \ \ 20 < t < 30 & \leftarrow \textit{OK here.} \\ -1 \ , \ if \ \ 30 < t & \textit{Not OK}: 2\pi. \end{cases}$$

For *t*'s larger than 30, though, we have one additional adjustment to do.

Since we do not want to mess up what we have done for any *t* with *t* < 30, but do want to make a new value for the right side for *t* > 30, the tool to use is the addition of a multiple of the unit step function

$$u_{30}(t)$$

As in the prior cases, adding a multiple of this function will not alter what we have previously achieved in matching the right and left sides. And, as previously, the multiplier should be such that we add in the new, desired value, -1 , at the same time as we remove the previously existing value, 2π. This means now that the desired multiplier is (-1 - (2π)). Finally, set

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t).$$

Evidently, for values of *t* > 30 the right side is always -1. Additionally, a little bit more thought will reveal to you that we now have a match on all intervals:

$$f(t) = \begin{cases} 4 \ , \ if \ \ 0 < t < 10 & \leftarrow \textit{OK here.} \\ -5 \ , \ if \ \ 10 < t < 20 & \leftarrow \textit{OK here.} \\ 2\pi \ , \ if \ \ 20 < t < 30 & \leftarrow \textit{OK here.} \\ -1 \ , \ if \ \ 30 < t & \leftarrow \textit{OK here.} \end{cases}$$

Really?? Previously I have left the *verification* to you. Now I shall actually reveal the *thinking*. Since the function *f* is piece-wise defined, we'll need to do some casework.

If 0 < *t* < 10, then

$$\begin{aligned} f(t) &= 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t) \\ &= 4(1) + (-5 - (4))(0) + (2\pi - (-5))(0) + (-1 - (2\pi))(0) \\ &= 4 \end{aligned}$$

If 10 < *t* < 20, then

$$\begin{aligned} f(t) &= 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t) \\ &= 4(1) + (-5 - (4))(1) + (2\pi - (-5))(0) + (-1 - (2\pi))(0) \\ &= 4 + (-5 - (4)) = -5 \end{aligned}$$

If $20 < t < 30$, then

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t)$$
$$= 4(1) + (-5 - (4))(1) + (2\pi - (-5))(1) + (-1 - (2\pi))(0)$$
$$= 4 + (-5 - (4)) + (2\pi - (-5)) = 2\pi$$

If $30 < t$, then

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t)$$
$$= 4(1) + (-5 - (4))(1) + (2\pi - (-5))(1) + (-1 - (2\pi))(1)$$
$$= 4 + (-5 - (4)) + (2\pi - (-5)) + (-1 - (2\pi)) = -1$$

There you have it. We will not repeat this sort of thing in the future. We'll leave the easy and tedious details to you.

Observe that the same core ideas allow us to write

$$f(t) = \begin{cases} -1 , & if \quad 0 < t < 3 \\ 2t-7 , & if \quad 3 < t < 5 \\ 3 , & if \quad 5 < t , \end{cases}$$

as

$$f(t) = -1\,u_0(t) + ((2t-7) - (-1))u_3(t) + (3 - (2t-7))u_5(t)$$
$$= -1 + (2t-6)u_3(t) + (10-2t)u_5(t)$$

very rapidly. Rapidly??? Yes, for in the previous example I belabored the reasoning and showed the adjustments while re-writing *f* again and again. Once one understands the structure and the process of 'reading down the line, left to right', the first line above may be written without much in the way of fuss -- even when the piece-wise defined function is not constant on the component sub-intervals. [You may check this on your own to see the telescoping magic recur.]

Once one has a piece-wise defined function written in terms of the unit steps, computing the Laplace transform is straightforward.

When we have *f* written as a simple linear combination of unit step functions, as in the first example, and as is always the case for step functions, the transform is nearly a no-brainer. Look. If

$$f(t) = 4u_0(t) + (-5 - (4))u_{10}(t) + (2\pi - (-5))u_{20}(t) + (-1 - (2\pi))u_{30}(t)$$
$$= 4 - 9u_{10}(t) + (2\pi+5)u_{20}(t) - (1+2\pi)u_{30}(t),$$

then, using linearity and the known transform for the generic unit step function, we have

$$\mathfrak{L}\{f(t)\}(s) = \frac{4}{s} - \frac{9e^{-10s}}{s} + \frac{(2\pi+5)e^{-20s}}{s} - \frac{(1+2\pi)e^{-30s}}{s}.$$

When the multipliers of the unit step functions are no longer mere constants, things become slightly more complicated. Consider the case of

$$f(t) = -1 + (2t-6)u_3(t) + (10-2t)u_5(t)$$

from our second piece-wise defined example. From the linearity of the transform, we have immediately that

$$\mathfrak{L}\{f(t)\}(s) = -\mathfrak{L}\{1\}(s) + \mathfrak{L}\{(2t-6)u_3(t)\}(s) + \mathfrak{L}\{(10-2t)u_5(t)\}(s).$$

The first summand above is an old friend, but to handle the second and third summands, we shall need to put (C) from Page 2 to work. [We could use (C′), but since Ross's text does not mention it, we shall avoid applying it directly.] To save you the labor of flipping back to the formula in question, here it is again:

(C) $$\mathfrak{L}\{g(t-a)u_a(t)\}(s) = e^{-as}\mathfrak{L}\{g(t)\}(s)$$

To make matters routine, we now shall denote the multipliers of the unit steps in the second and third summands by appropriate translates of functions so that we can use (C) above, thus:

$$\mathfrak{L}\{f(t)\}(s) = -\mathfrak{L}\{1\}(s) + \mathfrak{L}\{g(t-3)u_3(t)\}(s) + \mathfrak{L}\{h(t-5)u_5(t)\}(s).$$

$$\textit{where } g(t-3) = 2t-6 \ \textit{ and } \ h(t-5) = 10-2t.$$

You need to keep firmly in mind that the amount of the translation for *g* and *h* above is controlled by the index on the corresponding unit step function.

Using (C) now, we may write

$$\mathfrak{L}\{f(t)\}(s) = -\mathfrak{L}\{1\}(s) + e^{-3s}\mathfrak{L}\{g(t)\}(s) + e^{-5s}\mathfrak{L}\{h(t)\}(s).$$

$$\textit{where } g(t-3) = 2t-6 \ \textit{ and } \ h(t-5) = 10-2t.$$

To continue the computation, we need to have formulas for $g(t)$ and $h(t)$. These may be obtained algebraically by taking the formulas for $g(t - 3)$ and $h(t - 5)$ and replacing t uniformly using $t + 3$ to undo the translation by 3 and $t + 5$ to undo the translation by 5. Here we have it:

$$g(t) = g((t+3)-3) = 2(t+3)-6 = 2t$$

and

$$h(t) = h((t+5)-5) = 10-2(t+5) = -2t.$$

Thus,

$$\begin{aligned}\mathfrak{L}\{f(t)\}(s) &= -\mathfrak{L}\{1\}(s) + e^{-3s}\mathfrak{L}\{g(t)\}(s) + e^{-5s}\mathfrak{L}\{h(t)\}(s)\\ &= -\frac{1}{s} + e^{-3s}\mathfrak{L}\{2t\}(s) - e^{-5s}\mathfrak{L}\{2t\}(s)\\ &= -\frac{1}{s} + \frac{2e^{-3s}}{s^2} - \frac{2e^{-5s}}{s^2}.\end{aligned}$$

Here is a third, simple example, with the computation of the transform done without too much verbosity and only the essential bookkeeping:

$$f(t) = \begin{cases} 2t\ , & if\quad 0 < t < 3 \\ 6\ , & if\quad 3 < t \end{cases}$$
$$= 2t + (6-2t)u_3(t)$$

Observe that we immediately wrote *f* in terms of unit steps. Thus,

$$\begin{aligned} \mathfrak{L}\{f(t)\}(s) &= \mathfrak{L}\{2t\}(s) + \mathfrak{L}\{(6-2t)u_3(t)\}(s) \\ &= \frac{2}{s^2} + \mathfrak{L}\{g(t-3)u_3(t)\}(s),\ where\ g(t-3) = 6-2t \\ &= \frac{2}{s^2} + e^{-3s}\mathfrak{L}\{g(t)\}(s),\ \text{where } g(t) = g((t+3)-3) = -2t \\ &= \frac{2}{s^2} + e^{-3s}\mathfrak{L}\{-2t\}(s) = \frac{2}{s^2} - e^{-2s}\left(\frac{2}{s^2}\right). \end{aligned}$$

Finally, let us summarize things. To obtain the Laplace transform of a simple piece-wise continuous function, first write the function in terms of the unit step functions. This is done as a *telescoping sum* with *off-on switches* provided by unit steps whose indices have been chosen to make changes at appropriate times on the *t* line. If the function you started with is a step function, you should get a linear combination of unit steps. In this case, using linearity, the transform is simply the corresponding linear combination of the transforms of the component unit steps. When the original function is more complicated, item (C) needs to be used as in the last two examples, together with linearity.

Onward, ... eM toidI.