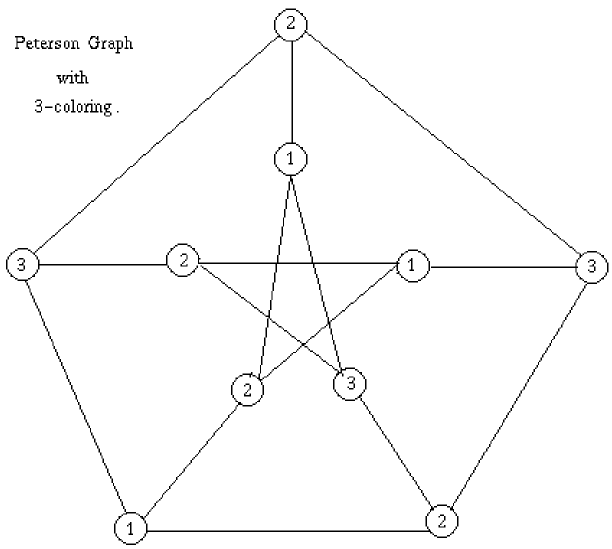
10.2 Determine the chromatic number of each of the following:

(a) the Peterson Graph, (b) the n-cube Q_n, (c) $W_n \cong C_n + K_1$.

Solution:

(a) Since the Peterson Graph, PG, has a 5-cycle, $\chi(PG) \geq 3$. Since PG is 3-regular and neither an odd cycle nor a complete graph, Brooks's Theorem implies that $\chi(PG) \leq 3$. Thus, $\chi(PG) = 3$. Of course if you get bored and have forgotten about Brooks's Theorem, you can always do a PG coloring with three colors yourself, as below:



(b) It turns out that it is not horribly difficult to prove by induction that your friendly n-cubes, defined recursively by $Q_1 = K_2$, and for $n \geq 2$, $Q_n = Q_{n-1} \times K_2$, are all nonempty bipartite graphs. Thus $\chi(Q_n) = 2$ for each $n \geq 1$.

(c) Now it's time to color the wheels of fortune, the W_n. Let us denote the K_1 vertex of W_n by w. Any coloring of W_n results in a coloring of the C_n contained within. This means that the vertices of C_n require at least 2 colors if n is even and at least 3 colors if n is odd. Since the vertex w is adjacent to all of the C_n vertices, we need an additional color for w. Hence, the chromatic number of W_n must be at least 3 if n is even and 4 if n is odd. On the other hand, a minimum coloring of C_n may be extended to a coloring of W_n by using one additional color. Thus, the chromatic number of W_n is at most 3 if n is even and 4 if n is odd. Consequently,

$$\chi(W_n) = \begin{cases} 3 & , \textit{if } n \textit{ is even}, \\ 4 & , \textit{if } n \textit{ is odd}. \end{cases}$$