4-3. Let f be a nonnegative measurable function. Show that $\int f = 0$ implies that f = 0 a.e.

Proof. Let $E_n = \{ x : f(x) > n^{-1} \}$ for each positive integer n. Clearly $\{ x : f(x) > 0 \} = \cup \{E_n : n \in \mathbb{N} \}$. Thus, to show that the measure of $\{ x : f(x) > 0 \}$ is zero and complete the proof, from Proposition 3-13, it suffices to show that each subset $E_n$ is of measure zero. To this end, fix $n \in \mathbb{N}$, and for each $k \in \mathbb{N}$, let $G_k = E_n \cap [-k,k]$. Since we have $E_n = \cup \{G_k : k \in \mathbb{N} \}$, we shall be finished once we show that the measure of $G_k$ is zero for each k. Fix k now, and to simplify the notation a tad, let $F = G_k$. Let $g(x) = n^{-1} \cdot \chi_F(x)$. Then g is a simple function with $g(x) \leq f(x)$ for x in the domain of f and nonzero on a set of finite measure. From the definition of the integral of f, and using that g is a simple function which is nonzero on a set with finite measure, it follows that

$$n^{-1} \cdot m(F) = \int g \leq \int f = 0.$$

Thus m(F) = 0, and we are finished.//