
Student Number:**Exam Number:**

Read Me First: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Be very careful. Read each question twice and provide an answer to exactly what is requested. Remember that what is illegible or incomprehensible is worthless. Good Luck!*

1. (15 pts.) (a) Suppose G_1 and G_2 are nontrivial graphs. What does it mean mathematically to say that G_1 and G_2 are isomorphic?? [This is really a request for the definition!]

(b) Give an example of two graphs G and H that have the degree sequence, have the same order and size, but are not isomorphic. Explain briefly how one can readily see that the graphs are not isomorphic.

2. (10 pts.) Use the ideas from the proof of Theorem 2.7, due to D. König, to construct a 3-regular graph G that contains $K_{1,3}$ as an induced subgraph. Show each stage of the construction. [This will not be the smallest 3-regular graph containing $K_{1,3}$ as an induced subgraph.]

3. (15 pts.) Use the Havel-Hakimi Theorem to show that the sequence

$$s: 5, 5, 4, 4, 3, 3$$

is the degree sequence of some graph. Then show there is a plane graph with this sequence as its degree sequence. [You may need to do a second drawing.]

4. (10 pts.) (a) Explain why \overline{C}_n is Hamiltonian for $n \geq 5$.

(b) Give an example of a simple connected graph G of order n with $\Delta(G) < n/2$ that is Hamiltonian. What does this tell us about Ore's Theorem?

5. (15 pts.) Each of the following propositions may be proved by induction. Prove exactly one of them after clearly indicating which you are proving.

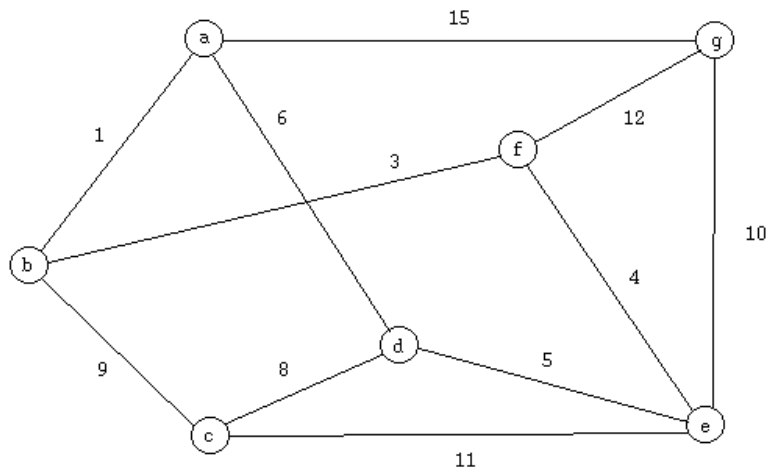
(a) Every nontrivial connected graph G has a spanning tree.
[Hint: If the order of the graph G is at least 3, Theorem 1.10 implies that G has a vertex v with $G - v$ connected.]

(b) If G is a connected plane graph of order n , size m , and having r regions, then $n - m + r = 2$. [Hint: Do induction on the size of G .]

6. (10 pts.) (a) What can you say about the order of any nontrivial digraph in which no two vertices have the same outdegree, but every two vertices have the same indegree? Why??

(b) What degree condition characterizes Eulerian digraphs? Is there an Eulerian digraph whose underlying graph is not Eulerian? [If "yes", show the simplest example you can think of.]

7. (10 pts.) Apply Kruskal's algorithm to find a minimum spanning tree in the weighted graph below. When you do this, list the edges in the order that you select them from left to right. What is the weight $w(T)$ of your minimum spanning tree T ?

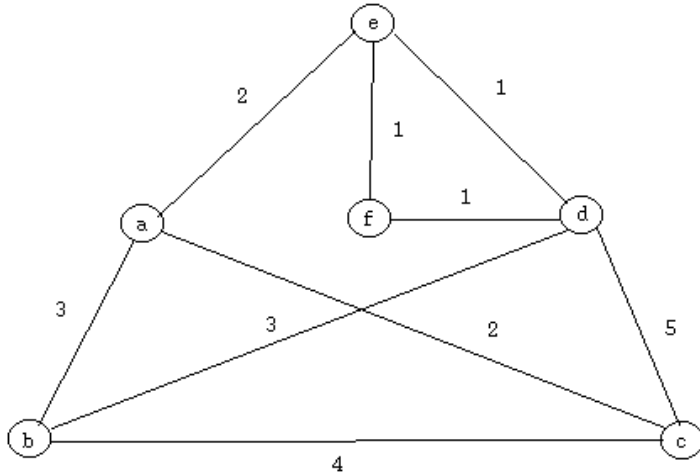


8. (15 pts.) (a) If G is a nontrivial graph, how is $\kappa(G)$, the vertex connectivity of G , defined?

(b) If G is a nontrivial graph and v is a vertex of G , $\kappa(G - v) \geq \kappa(G) - 1$. Provide the simple proof for this.

(c) If G is a Hamiltonian graph, what can you say about $\kappa(G)$? Why??

9. (15 pts.) Find a minimum spanning tree for the weighted graph below by using only Prim's algorithm and starting with the vertex a. When you do this, list the edges in the order that you select them from left to right. What is the weight $w(T)$ of your minimum spanning tree T ?

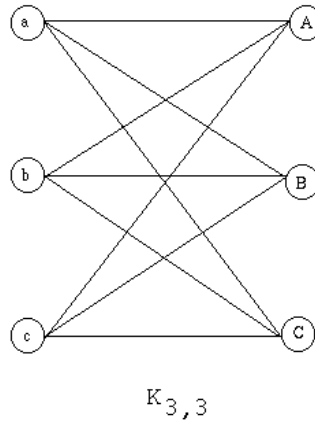


10. (10 pts.) (a) What is the chromatic number of any nontrivial tree? Why?

(b) Give an example of a nonplanar graph G with $\chi(G) = 4$.

11. (10 pts.) (a) Suppose that G is a bipartite graph with partite sets U and W with $|U| \leq |W|$. What does it mean to say that U is neighborly?

(b) $K_{3,3}$ is given below with partite sets $U = \{a, b, c\}$ and $W = \{A, B, C\}$. Give a connected bipartite subgraph of $K_{3,3}$ with the same partite sets but with U failing to be neighborly. Explain why your example satisfies the requirements.



12. (15 pts.) (a) State Kuratowski's Theorem that characterizes planar graphs.

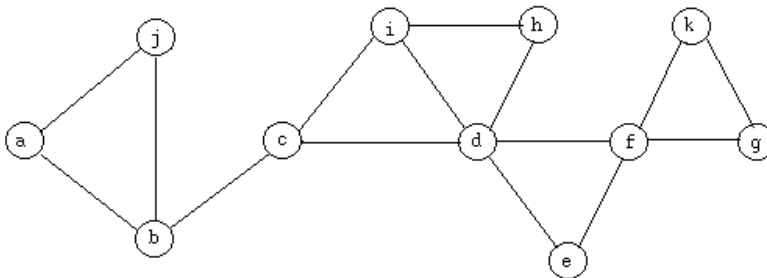
(b) Without using Kuratowski's Theorem, explain briefly how one can see that K_5 is not planar.

(c) Without using Kuratowski's Theorem, explain briefly how one can see that $K_{3,3}$ is not planar.

13. (25 pts.) (a) (5 pts.) What is a bridge? [Hint: Original definition please.]

(b) (5 pts.) What is a cut-vertex?? [Definition, please.]

(c) (15 pts.) For the graph G below, determine the cut-vertices, bridges, and blocks of G . List the cut-vertices and bridges in the appropriate places, and provide carefully labelled sketches of the blocks.



Cut-vertices:

Bridge(s):

Block(s):

14. (25 pts.) (a) (5 pts.) What is a network, N ?? [Hint: It would be nice to see Gould's definition.]

(b) (5 pts.) What is a legal (or feasible) flow in a network N ? [Hint: Definition.]

(c) (15 pts.) Obtain a maximum flow f in the network below, and verify the flow is a maximum by producing a set of vertices S that produces a minimum cut. Check that the total capacity of that cut is the same as the value of your max flow.

