
General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless. Show me all the magic on the page.

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1. (25 pts.) Provide brief answers to each of the following:

(a) What is the order and size of the graph $K_{9,13}$?? Say which is which unequivocally.

The complete bipartite graph $K_{9,13}$ has order 22 and size 117.

[Recall that the vertex set is partitioned into a 9 element set and a 13 element set?? Every vertex in the 9 element set is adjacent to every vertex in the 13 element set???]

(b) The degree of each vertex of a certain graph of order 12 and size 31 is either 4 or 6. How many vertices of degree 6 are there?

Let n denote the number of vertices of degree 6. Then the 1st Theorem of Graph Theory implies that we have

$$6n + 4(12 - n) = (2)(31).$$

Solving this yields n = 7.

(c) Suppose G is a connected non-trivial graph and u and v are two vertices with d(u,v) = 47. What is the order of the smallest connected subgraph H of G that contains u and v? What can you say about the diameter of G?

The order of the smallest connected subgraph H of G that contains u and v is the order of the u-v geodesic that has length 47, the distance from u to v. Clearly this geodesic is of order 48.

Evidently, the diameter of G is at least 47. You could write this as follows: $\mathrm{diam}(G) \geq 47$.

(d) Suppose that G is a graph of order 25 and size 99. From this information, we know that any trail in G can be no longer than what number l? Provide the best upper bound.

Sorry about the notation, the lower case L. Trails have to have distinct edges. Hence the best bound with the current information is l = 99.

(e) Suppose that G is a graph of order 25 and size 99. From this information, we know that any path in G can be no longer than what number l? Provide the best upper bound.

Paths have to have distinct vertices. The longest possible would be with 25 vertices and have length 24.

2. (20 pts.) Provide mathematical definitions for each of the following terms.

(a) A graph G: A graph G consists of a finite nonempty set V of vertices and a set E of 2-element subsets of V called edges.

[For convenience, we frequently write V as V(G) and E as E(G) when we are dealing with more than one graph as a time.]

(b) Subgraph: A graph H is called a subgraph of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

(c) Spanning Subgraph: A subgraph H of a graph G is said to be a spanning subgraph if the vertex set of H is the same as the vertex set of G, that is, V(H) = V(G).

(d) Bipartite Graph: A graph G is a bipartite graph if there are nonempty subsets U and W of V(G) with $U \cup W = V(G)$, $U \cap W = \phi$, and each edge of G joins a vertex from U and a vertex from W.

[The sets U and W are called partite sets.]

(e) Diameter: The greatest distance between any two vertices of a connected graph G is called the diameter of G and is denoted diam(G). So diam(G) = max $\{ d(u,v): u,v \in V(G) \}$.

3. (5 pts.) List the r-regular graphs of order 5 for all possible values of r. They are all old friends.

The r-regular graphs of order 5 are the empty graph of order 5 $\overline{K}_5$ with r = 0, the 5-cycle C_5 with r = 2, and the complete graph K_5 with r = 4. [I'll accept sketches, but I really wanted the "names".]

4. (10 pts.) Use the Havel-Hakimi Theorem to construct a graph with degree sequence

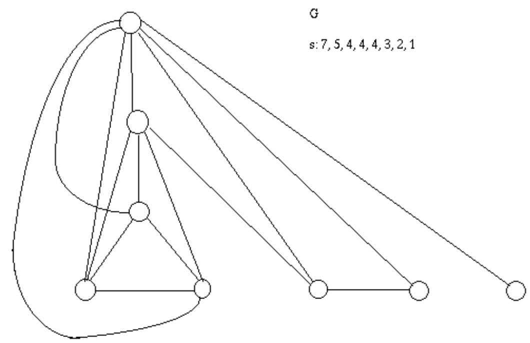
s: 7,5,4,4,4,3,2,1

s_1: 4,3,3,3,2,1,0

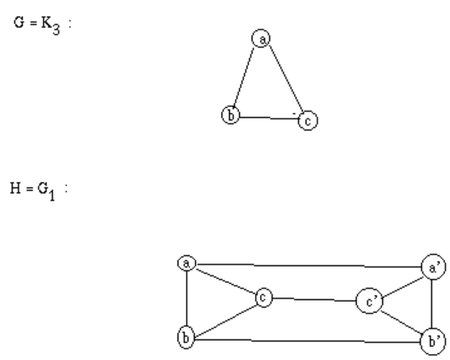
s_2: 2,2,2,1,1,0

s_3: 1,1,1,1,0

[Graphic]

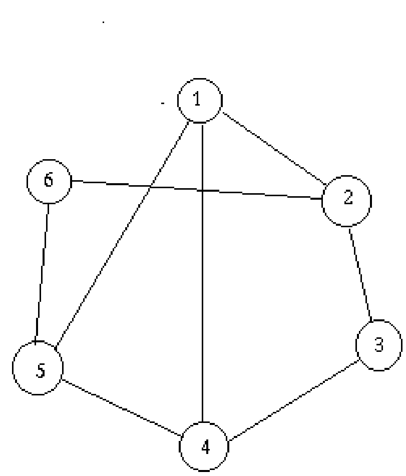


5. (10 pts.) Use the ideas from the proof of Theorem 2.7, to construct a 3-regular graph G that contains K_3 as an induced subgraph. Show each stage of the construction.



6. (5 pts.) Sketch a graph G that has the following adjacency matrix:

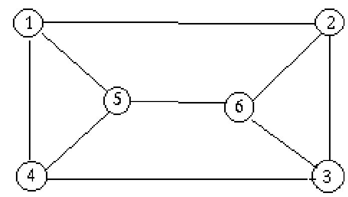
$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$



7. (5 pts.) Construct a 3-regular graph G of minimum order that contains C_4 as an induced subgraph. [Use the ideas of Paul Erdos and Paul J. Kelly.]

Since $\delta(C_4) = 2$ and $\Delta(C_4) = 2$, we need at least one vertex. There is not a 3-regular graph of order 5. So 6 vertices will be the best that we can do.

G



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8. (10 pts.) Prove exactly one of the following propositions. Indicate clearly which you are demonstrating.

(a) If G is a non-trivial graph, then there are distinct vertices u and v in G with deg(u) = deg(v).

(b) If G is a graph of order n and deg(u) + deg(v) $\geq$ n - 1 for each pair of non-adjacent vertices u and v, then G is connected.

// Review??
(a): Theorem 2.14, page 51.

(b): Theorem 2.4, page 34.

Proofs were also done in class. These varied somewhat from those of the text.

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9. (10 pts.) (a) Suppose G is a bipartite graph of order at least 5. Prove that the complement of G is not bipartite. [Hint: At least one partite set has three elements. Connect the dots?]

Suppose that G is a bipartite graph of order at least 5 with partite sets U and W. At least one of U and W has at least 3 elements. Suppose without loss of generality, $|W| \geq 3$. Label three of the members of W with u,v, and w. Since these vertices are in the same partite set of G, none of these three vertices is adjacent to any other of the three. Thus,

$$uv,\ vw,\ uw \in E(\overline{G}) \Rightarrow \overline{G} \text{ contains a } 3\text{-cycle}.$$

Thus, the complement of G is not bipartite. [Problem 1.25?]

(b) Display a bipartite graph G of order 4 and its bipartite complement. Label each appropriately and give partite sets for each bipartite graph.
//There are a multitude of examples. See me if you need help with this! This reveals why we want $|V(G)| \geq 5$ in (a).