General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless.

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1. (25 pts.) (a) What is a bridge?? [Yes, a definition is required.]

If G is a nontrivial connected graph, then an edge e of G is a bridge if G - e is not connected. In the case of an arbitrary graph G, say with components G_i for i = 1, ..., m, an edge e of G is a bridge if G_k - e is disconnected where G_k is the component containing e.

(b) Give an example of a graph of order n and size n - 1 that is not a tree.

Evidently, $G = K_1 \cup C_{n-1}$ for $n \geq 4$ will do the job. Likely as not you sketched $G = K_1 \cup C_3$.

(c) A certain tree of order n has only vertices of degree 3 and degree 1. How many degree 3 vertices does the tree have?

Let x denote the number of vertices of degree 3 that the tree has. Then the 1st Theorem of Graph Theory implies that $3x + 1(n - x) = 2(n - 1)$. Solving this yields $x = (n - 2)/2$.

(d) If G is a nonseparable graph with order at least 3, what is the best estimate that you provide for $\delta(G)$? Why??

A graph of order at least 3 is nonseparable if, and only if every pair of vertices lie on a common cycle. Consequently, we can say $\delta(G) \geq 2$. We cannot do better than 2 generally since n-cycles are 2-regular.

(e) Let T be a tree of order $n \geq 2$ with $k = \Delta(G)$. Let n_i be the number of vertices of degree i for $1 \leq i \leq k$. Then

$$\sum_{i=1}^{k} n_i = n \quad \text{and} \quad 2(n-1) = \sum_{i=1}^{k} i n_i \ .$$

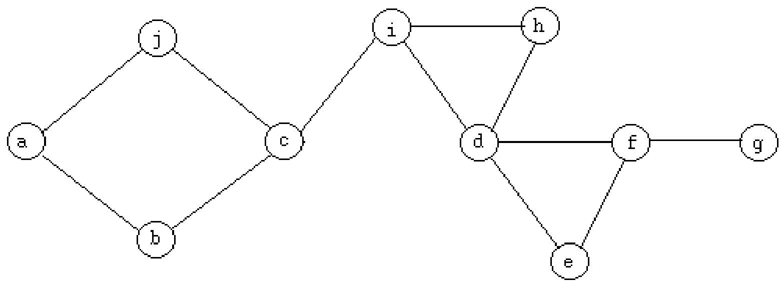
Show how to obtain a formula for the number of end-vertices of T.

By replacing n in the second equation with the summation in the first equation, it follows that

$$2\left(\sum_{i=1}^{k} n_i\right) - 2 = \sum_{i=1}^{k} i n_i \Rightarrow n_1 = 2 + \sum_{i=2}^{k} i n_i - \sum_{i=2}^{k} 2 n_i$$

$$\Rightarrow n_1 = 2 + \sum_{i=2}^{k} (i-2) n_i$$

2. (15 pts.) For the graph G below, determine the cut-vertices, bridges, and blocks of G. List the cut-vertices and bridges in the appropriate places, and provide carefully labelled sketches of the blocks.



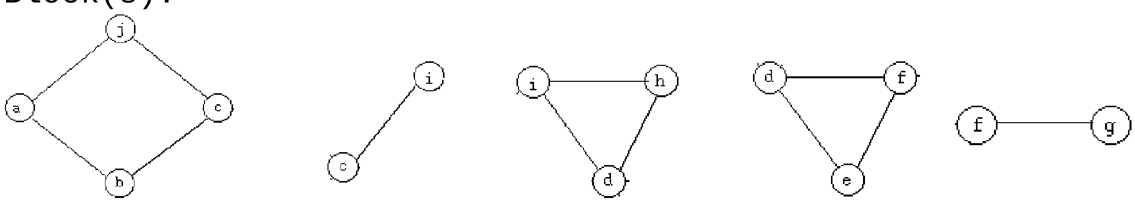
Cut-vertices:

c, d, f, and i

Bridge(s):

ci and fg

Block(s):



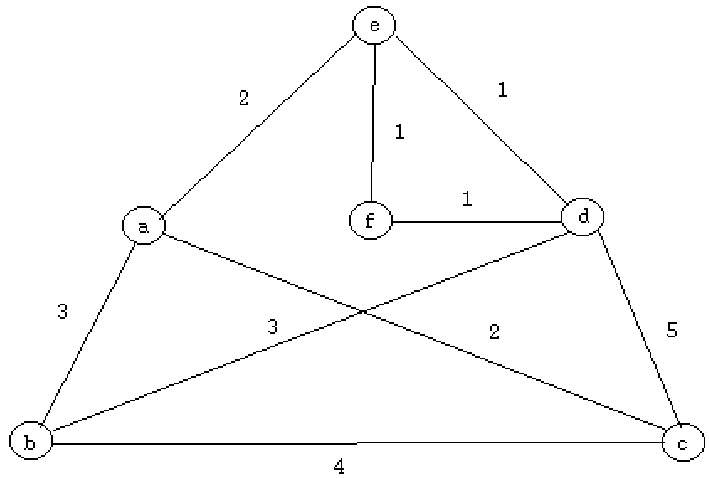
3. (10 pts.) Below, provide a proof by induction on the order of the graph G that every nontrivial connected graph G has a spanning tree. [Hint: If the order of the graph G is at least 3, Theorem 1.10 implies that G has a vertex v with G - v connected.] //

Let S(k) be the following assertion: "For any graph G, if G is connected with k vertices, then G has a spanning tree T."

If G is a connected graph of order 2, then G is K_2 which is a tree and there is nothing to show. Thus S(2) is true, and we get a basis for the induction with no real work.

To deal with the induction step, we must provide a proof of the proposition $(\forall k \geq 2)(S(k) \Rightarrow S(k+1))$. To show this, we let $k \geq 2$ be fixed and arbitrary. We need to show for this k that $S(k) \Rightarrow S(k+1)$. Suppose that S(k) is true. To show that S(k+1) follows, we may assume as true the hypothesis of S(k+1), namely that we are dealing with G, an arbitrary connected graph with k+1 vertices. We need to show that G has a spanning tree. Since G has at least 3 vertices, Theorem 1.10 implies that there are two vertices in G, say u and v, with G - u and G - v connected. We shall focus on G - v. G - v is a connected graph with $k \geq 2$ vertices. From the induction hypothesis, S(k), with G - v replacing G, it follows that G - v has a spanning tree, T_0 say. Since G is connected, v is adjacent to at least one of the vertices of G - v. Pick such a vertex, say w, and let $T = (V(G), E(T_0) \cup \{vw\})$. Then T is a spanning tree for G. Since G was an arbitrary connected graph of order k+1, S(k+1) follows. Thus, $S(k) \Rightarrow S(k+1)$. Since k was arbitrary, we have $(\forall k \geq 2)(S(k) \Rightarrow S(k+1))$. We may now apply the principle of induction. [Hello, modus ponens.]

5. (10 pts.) Apply Kruskal's algorithm to find a minimum spanning tree in the weighted graph below. When you do this, list the edges in the order that you select them from left to right. What is the weight w(T) of your minimum spanning tree T?



There are several different correct solutions. First, in some order you will take two of the three edges from the "1" 3-cycle: de, ef, fd.

Then you will need both "2" edges in some order: ae and ac.

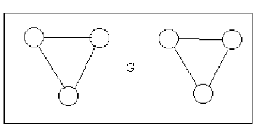
Then exactly one of the two "3" edges: ab or bd.

For any of the 6 possible trees T, w(T) = 9.

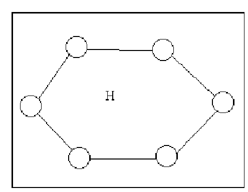
6. (15 pts.) (a) Suppose G_1 and G_2 are nontrivial graphs. What does it mean mathematically to say that G_1 and G_2 are isomorphic?? [This is really a request for the definition!]

Two graphs G_1 and G_2 are isomorphic if there is a bijection $\phi: V(G_1) \to V(G_2)$ such that uv ε $E(G_1)$ if, and only if $\phi(u)\phi(v)$ ε $E(G_2)$.

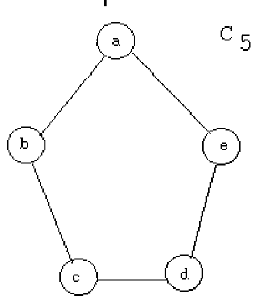
(b) Sketch two graphs G and H that have the degree sequence s: 2, 2, 2, 2, 2, 2 and have the same order and size, but are not isomorphic. Explain briefly how one can readily see that the graphs are not isomorphic.



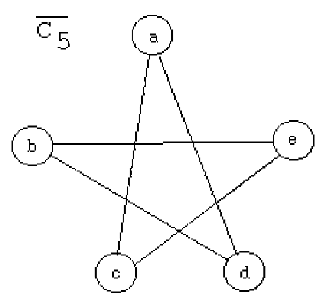
Since G is not connected and H is connected, G and H cannot be isomorphic. Why? Any graph isomorphic to H must be connected.



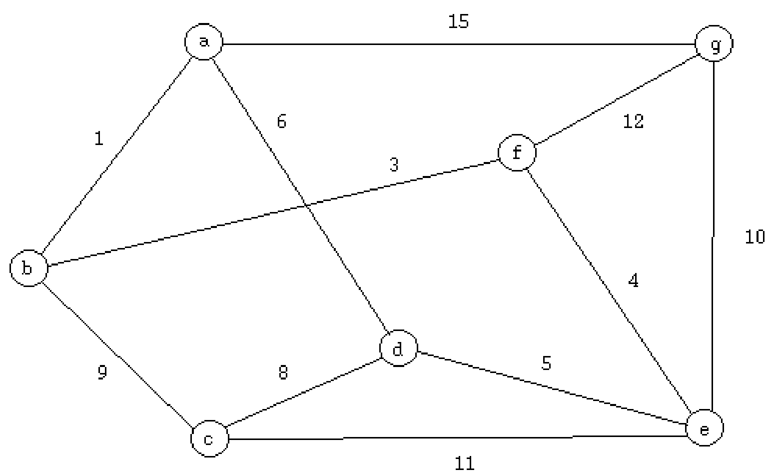
(c) Explicitly realize C_5 and its complement below. [You may provide carefully labelled sketches.] Next, explicitly define an isomorphism from C_5 to its complement that reveals that C_5 is self-complementary.



Define ϕ from C_5 to its complement by $\phi(a) = a$, $\phi(b) = c$, $\phi(c) = e$, $\phi(d) = b$, and $\phi(e) = d$. ϕ is a bijection, and its easy to check uv is an edge in C_5 exactly when $\phi(u)\phi(v)$ is an edge in the complement of C_5. [There are only 5 edges to check.] How many isomorphisms are there??



7. (10 pts.) Find a minimum spanning tree for the weighted graph below by using only Prim's algorithm and starting with the vertex g. When you do this, list the edges in the order that you select them from left to right. What is the weight w(T) of your minimum spanning tree T?



Beginning with vertex g, you should obtain edges in the following order:

ge, ef, fb, ba, ed,

dc.

There is only one spanning tree, and its weight is

w(T) = 31.

You can check the edges, but not their order using Kruskal's algorithm.

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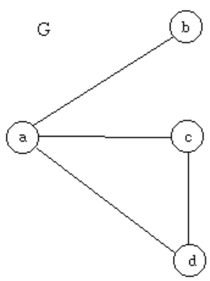
8. (15 pts.) (a) If G is a nontrivial graph, how is $\kappa(G)$, the vertex connectivity of G, defined?

If G is a complete graph of order n, then $\kappa(G) = n - 1$. Otherwise, G has a vertex-cut. In this case, $\kappa(G) = k$ where k is the cardinality of a minimum vertex-cut.

(b) If G is a nontrivial graph, it is not true generally that if v is an arbitrary vertex of G, then either $\kappa(G - v) = \kappa(G) - 1$ or $\kappa(G - v) = \kappa(G)$. Give a simple example of a connected graph G illustrating this. [A carefully labelled drawing with a brief explanation will provide an appropriate answer.]



Evidently, $G \cong K_1 + (K_1 \cup K_2)$. Since G is connected and a is a cut-vertex of G, $\kappa(G) = 1$. Note, however, $G - b \cong K_3$, and thus $\kappa(G - b) = 2$, not 0 or 1.

(c) Despite the example above, if G is a nontrivial graph and v is a vertex of G, $\kappa(G - v) \geq \kappa(G) - 1$. Provide the simple proof for this.

Proof: Let v be an arbitrary vertex of G. If G is a complete graph of order k, then G - v is complete of order k - 1. Thus, the conclusion follows from definition of κ. So suppose G is not a complete graph. Then G - v is not a complete graph, too. Then either $\kappa(G - v) \geq \kappa(G) - 1$ or $\kappa(G - v) < \kappa(G) - 1$. If $\kappa(G - v) < \kappa(G) - 1$, then there is a subset U_0 of the vertex set of G - v with $|U_0| = \kappa(G - v)$ and $(G - v) - U_0$ disconnected. But then $U = U_0 \cup \{v\}$ is a vertex-cut of G for which $|U| = |U_0| + 1 < \kappa(G)$, an impossibility. Hence we must have $\kappa(G - v) \geq \kappa(G) - 1$.//