General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless.

1. (25 pts.) (a) What is a bridge?? [Yes, a definition is required.]

(b) Give an example of a graph of order n and size n - 1 that is not a tree.

(c) A certain tree of order n has only vertices of degree 3 and degree 1. How many degree 3 vertices does the tree have?

(d) If G is a nonseparable graph with order at least 3, what is the best estimate that you provide for $\delta(G)$? Why??

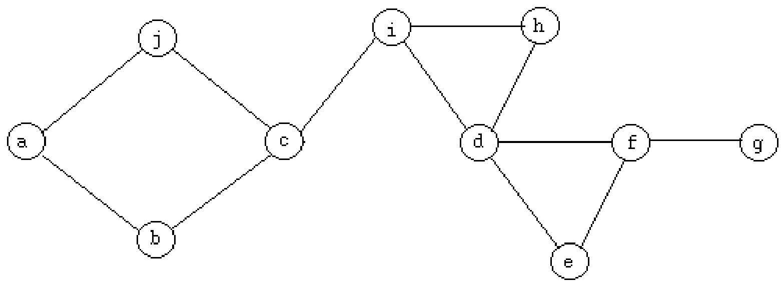
(e) Let T be a tree of order $n \geq 2$ with $k = \Delta(G)$. Let n_i be the number of vertices of degree i for $1 \leq i \leq k$. Then

$$\sum_{i=1}^{k} n_i = n \quad \text{and} \quad 2(n-1) = \sum_{i=1}^{k} i n_i \ .$$

Show how to obtain a formula for the number of end-vertices of T.

2. (15 pts.) For the graph G below, determine the cut-vertices, bridges, and blocks of G. List the cut-vertices and bridges in the appropriate places, and provide carefully labelled sketches of the blocks.

Cut-vertices:

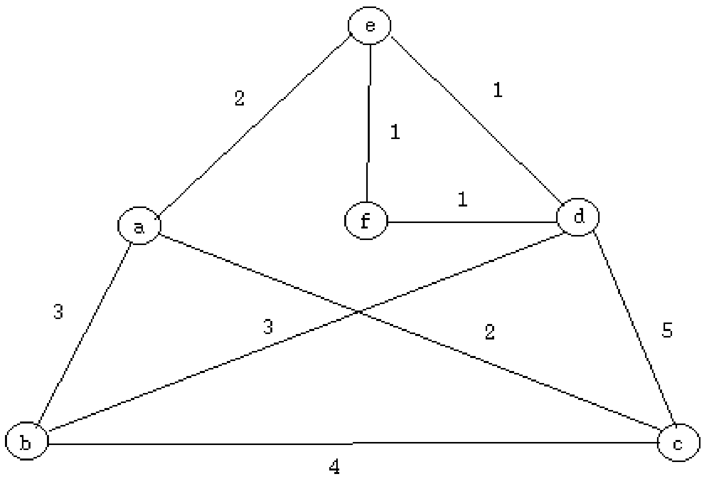


Bridge(s):

Block(s):

3. (10 pts.) Below, provide a proof by induction on the order of the graph G that every nontrivial connected graph G has a spanning tree. [Hint: If the order of the graph G is at least 3, Theorem 1.10 implies that G has a vertex v with G - v connected.]

5. (10 pts.) Apply Kruskal's algorithm to find a minimum spanning tree in the weighted graph below. When you do this, list the edges in the order that you select them from left to right. What is the weight w(T) of your minimum spanning tree T?

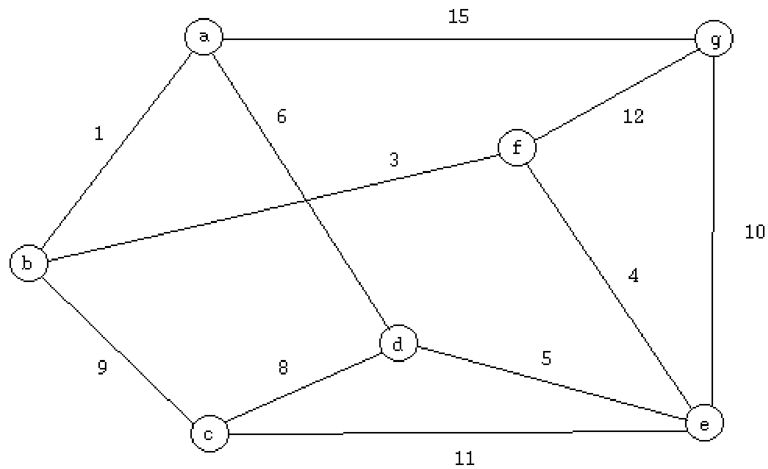


6. (15 pts.) (a) Suppose G_1 and G_2 are nontrivial graphs. What does it mean mathematically to say that G_1 and G_2 are isomorphic?? [This is really a request for the definition!]

(b) Sketch two graphs G and H that have the degree sequence s: 2, 2, 2, 2, 2, 2 and have the same order and size, but are not isomorphic. Explain briefly how one can readily see that the graphs are not isomorphic.

(c) Explicitly realize C_5 and its complement below. [You may provide carefully labelled sketches.] Next, explicitly define an isomorphism from C_5 to its complement that reveals that C_5 is self-complementary.

7. (10 pts.) Find a minimum spanning tree for the weighted graph below by using only Prim's algorithm and starting with the vertex g. When you do this, list the edges in the order that you select them from left to right. What is the weight w(T) of your minimum spanning tree T?



8. (15 pts.) (a) If G is a nontrivial graph, how is $\kappa(G)$, the vertex connectivity of G, defined?

(b) If G is a nontrivial graph, it is not true generally that if v is an arbitrary vertex of G, then either $\kappa(G - v) = \kappa(G) - 1$ or $\kappa(G - v) = \kappa(G)$. Give a simple example of a connected graph G illustrating this. [A carefully labelled drawing with a brief explanation will provide an appropriate answer.]

(c) Despite the example above, if G is a nontrivial graph and v is a vertex of G, $\kappa(G - v) \geq \kappa(G) - 1$. Provide the simple proof for this.