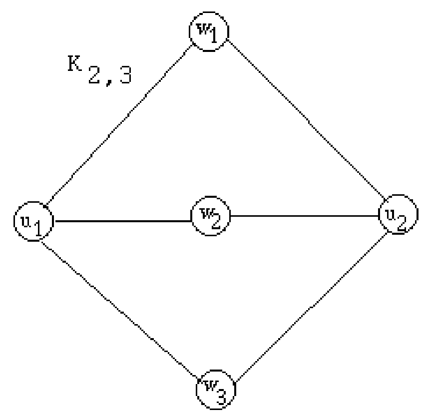
General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless.

1. (15 pts.) (a) If G is a connected planar graph with 6 vertices, what can you tell me about the size of G?

If m is the size of G, then $m \le 3(6) - 6 = 12$, the size of a maximal planar graph with 6 vertices.

(b) For which pairs of integers r and s is $K_{r,s}$ planar and for which is $K_{r,s}$ nonplanar? Provide a brief explanation and/or a plane graph drawing, as appropriate, to deal with the various situations.



If r = 1 or s = 1, then $K_{r,s}$ is a tree, and thus planar. If both r and s are at least 3, then $K_{r,s}$ contains $K_{3,3}$ and is not planar. What is left is the case where one of r or s is 2 and the other is at least 2. Without loss of generality, we may take $r = 2$ and $s \ge 2$. In this case, $K_{r,s}$ is planar. To see this, let U be the partite set with r elements and W be the partite set with s members. $K_{r,s}$ may be displayed as a plane graph by lining up the vertices from W vertically in a line, placing one element of U to the left of the W vertices and the other to the right, and then connecting the dots in the obvious way. $K_{2,3}$ above and to the right is typical.

2. (10 pts.) (a) Suppose that G is a bipartite graph with partite sets U and W with $|U| \le |W|$. What does it mean to say that U is neighborly?

U is *neighborly* if for each nonempty subset X of U, $|X| \le |N(X)|$, where

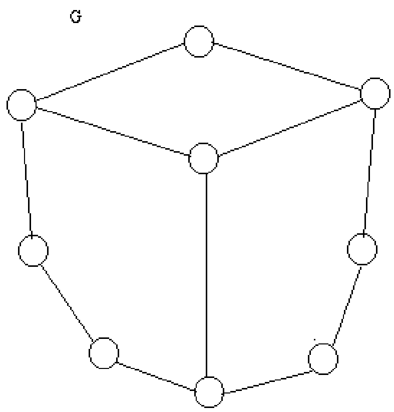
$$N(X) = \bigcup_{v \in X} N(v)$$

is the set of vertices in G adjacent to vertices of X. Here. of course, $N(v) = \{ w \in V(G): vw \in E(G)\}$.

(b) Recall that your friendly n-cubes are defined recursively by $Q_1 = K_2$, and for $n \ge 2$, $Q_n = Q_{n-1} \times K_2$. Do these friendly bipartite graphs have perfect matchings?? Explain briefly.

But of course they do. The key observation is that our friendly n-cubes are n-regular and thus have a neighborly partite set to aid matters. [No, you do not have to show me that!]

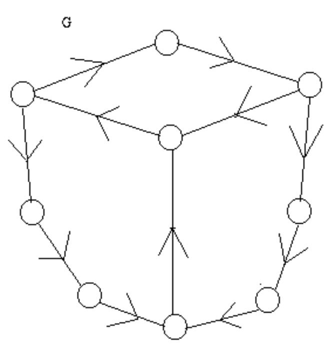
3. (15 pts.) (a) Show that the graph below has a strong orientation by assigning a direction to each edge so that the resulting digraph is strong.



You'll find one such below the given graph. The key: Just take a leisurely walk and keep track of the direction you are walking so that you don't violate the orientation you are creating, and hope there are no bridges.

(b) The graph to the left has many strong orientations. Does the graph have an Eulerian orientation? Explain briefly.

No. Since G has an odd vertex v, every orientation will have id(v) ≠ od(v). Thus, there are no Eulerian orientations.



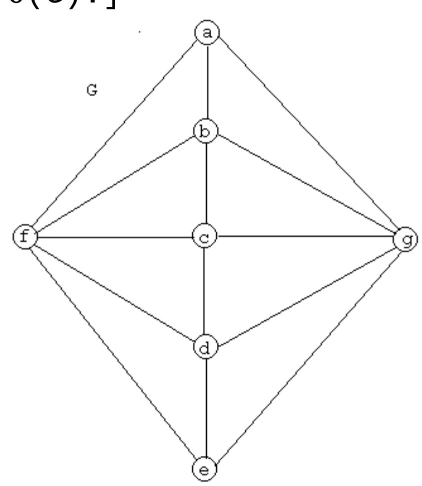
A Strong Orientation for G

(c) If you were asked to give me an example of a connected graph which has no orientations that are strong, what feature(s) would you include in your example to ensure that the example satisfies the requirement? Why??

Be sure to include a bridge since a connected graph has a strong orientation if, and only if it has no bridges.

4. (10 pts.) Theorem 5.17, a corollary of sorts to Menger's Theorem allows you to deal with the vertex connectivity of the graph below easily. Explain briefly. [Hint: Look north-south as well as east-west after considering $\delta(G)$.]

Since $\kappa(G) \le \lambda(G) \le \delta(G) = 3$, G is at most 3-connected. If you scan the graph carefully, you will observe that there are at least 3 internally disjoint paths between every pair of vertices. Consequently, Theorem 5.17 implies that the graph to the right is 3-connected. [Yes, to really "prove" this would be a fairly boring task. Here you are permitted to wave your hands.]

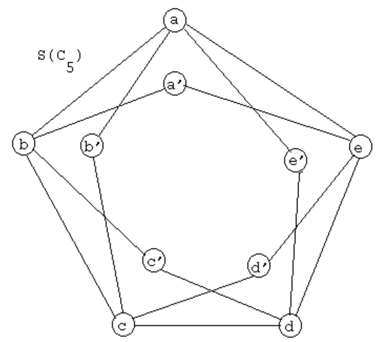


5. (15 pts.) (a) What is a clique?

A *clique* in a graph G is a complete subgraph of G, i.e., a subgraph isomorphic to a K_n for some $n \geq 1$.

(b) Sketch the shadow graph $S(C_5)$ of a generic 5-cycle below. What is $\chi(S(C_5))$??

Plainly any coloring of C_5 may be extended to a coloring of $S(C_5)$ with the same number of colors by assigning each primed vertex the same color as its un-primed twin. Thus, we have $\chi(S(C_5)) \leq \chi(C_5) = 3$. On the other hand, C_5 is a subgraph of $S(C_5)$. Thus, $\chi(S(C_5)) \geq \chi(C_5) = 3$. So $\chi(S(C_5)) = 3$.



[What is going on here is true more generally, right?]

(c) How is the Grötzsch graph, which we will denote by G here, obtained from the shadow graph of Part (b) above?? It turns out that $\omega(G) = 2$ and $\chi(G) = 4$. What is the significance of this??

The Grötzsch graph is obtained from $S(C_5)$ above by adding a vertex that is adjacent only to the primed vertices. This ensures that the chromatic number of the Grötzsch graph is 4 while not creating any triangles since the primed vertices are independent. The significance of this is that we now can see how to recursively build graphs G with the difference $\chi(G) - \omega(G)$ as big as we please. Big cliques are not required for lots of colors.

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6. (10 pts.) Prove, by induction on the size of the graph, that if G is a connected plane graph of order n, size m, and having r region, then (*) $n - m + r = 2$.

As a basis for the induction, we note that if m = 0, 1, or 2, then we are dealing with a tree so that n = 1, 2, or 3 respectively, and there is only 1 region. For these values, equation (*) is valid. Suppose that $m > 3$, and for the induction hypothesis, that (*) is true for connected plane graphs having fewer than m edges. Let G be a connected plane graph with m edges and n vertices. If G is a tree, there is only one region and $n = m + 1$. In this case equation (*) is true trivially. Thus, suppose G is not a tree. Then G has a cycle. Let e be an edge lying on a cycle. The graph G - e is plainly a connected plane graph with m - 1 edges and n vertices. The removal of the edge e decreases the number of regions created by one. Thus, from the induction hypothesis, applied to the graph G - e, we see that (*) is true with n replaced by n, m replaced by m - 1, and r replaced by r - 1. Thus $n + (m - 1) - (r - 1) = 2$, which implies that $n + m - r = 2$ by doing the arithmetic. Thus, ... yadda, yadda, yadda. [Write the formal incantation to finish the proof.]//

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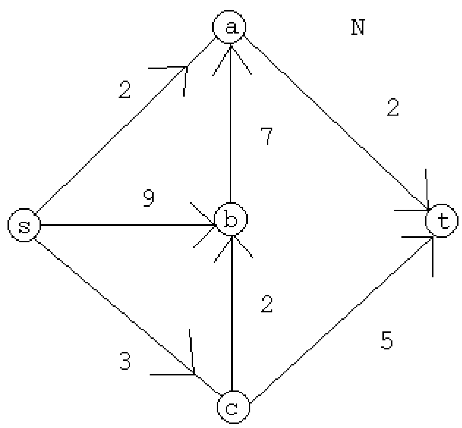
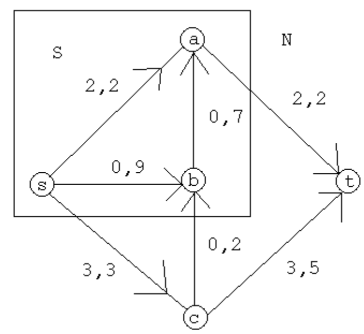
7. (15 pts.) (a) What is a legal (or feasible) flow in a network N ? [Hint: Definition.]// If N = (V, E, s, t, c) is a network, then a legal flow on N is a function $f:E \to [0,\infty)$ such that $f(e) \le c(e)$ for each edge $e \in E$, and

$$\sum_{e \in in(v)} f(e) = \sum_{e \in out(v)} f(e)$$

for each $v \in V - \{s,t\}$. Here, $c:E \to [0,\infty)$ is the capacity function.

(b) Obtain a maximum flow f in the network below, and verify the flow is a maximum by producing a set of vertices S that produces a minimum cut. Check that the total capacity of that cut is the same as the value of your max flow.





N with a Maximum Flow
val(f) = 5 and c(S) = c((s,c)) + c((a,t)) = 5.

Start with the zero flow.

Labelling Algorithm with labels over "reached" vertices.
[We underline reached, labelled vertices to keep track of the vertex we are scanning from. Once we are done, we copy the vertex to the scanned (from) list below. We choose t first if it is in the neighborhood. Once we have an f-augmenting path and slack in hand, we update the network diagram. The final update is to the upper right.]

	* s+ s+ s+ a+					
1st:	R: $\underline{s}$, $\underline{a}$, b, c, t	Path :	s —— a —— t			
	S: s, a	Slack:	2	2		$\lambda = 2$
	* s+ s+ b+ c+					
2nd:	R: $\underline{s}$, $\underline{b}$, $\underline{c}$, a, t	Path :	s —— c —— t			
	S: s, b, c	Slack:	3	5		$\lambda = 3$
	* s+ b+					
3rd:	R: $\underline{s}$, $\underline{b}$, $\underline{a}$					
	S: s, b, a	R = S = {s,b,a}				
		Halt. Last flow is a max.				

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8. (10 pts.) Which complete bipartite graphs $K_{r,s}$ are Hamiltonian and which are not? Explain briefly. [Hint: When can you use Dirac? What is the well-known necessary condition?]

First, all the complete bipartite graphs are connected. Since $K_{1,1}$ is essentially K_2, which has no cycles, $K_{1,1}$ is not Hamiltonian. For $r \ge 2$, if $r = s$, then the order of the graph is $2r$ and $\delta(K_{r,r}) = r$. Consequently, we can use Dirac's Theorem to see that $K_{r,s}$ is Hamiltonian when $r = s \ge 2$. When $r \ne s$, $K_{r,s}$ is not Hamiltonian. To see this, we may assume without loss of generality that $r < s$. Then, if U is the r partite set of vertices of $K_{r,s}$, then $k(K_{r,s} - U) = s > r = |U|$.//