General directions: Read each problem carefully and do exactly what is requested. Show all your work neatly. Use complete sentences and use notation correctly. Make your arguments and proofs as complete as possible. Remember that what is illegible or incomprehensible is worthless.

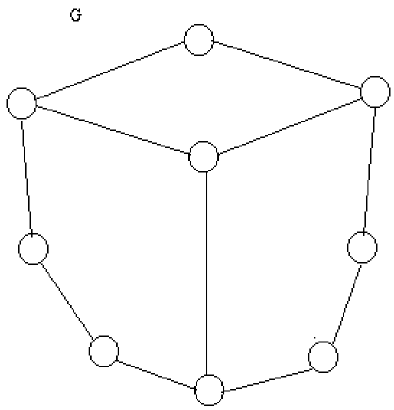
1. (15 pts.) (a) If G is a connected planar graph with 6 vertices, what can you tell me about the size of G?

(b) For which pairs of integers r and s is $K_{r,s}$ planar and for which is $K_{r,s}$ nonplanar? Provide a brief explanation and/or a plane graph drawing, as appropriate, to deal with the various situations.

2. (10 pts.) (a) Suppose that G is a bipartite graph with partite sets U and W with $|U| \leq |W|$. What does it mean to say that U is neighborly?

(b) Recall that your friendly n-cubes are defined recursively by $Q_1 = K_2$, and for $n \geq 2$, $Q_n = Q_{n-1} \times K_2$. Do these friendly bipartite graphs have perfect matchings?? Explain briefly.

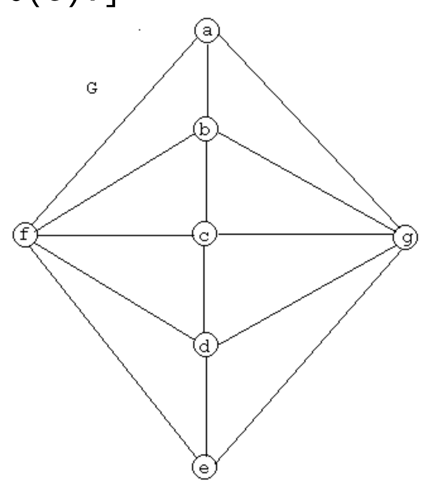
3. (15 pts.) (a) Show that the graph below has a strong orientation by assigning a direction to each edge so that the resulting digraph is strong.



(b) The graph to the left has many strong orientations. Does the graph have an Eulerian orientation? Explain briefly.

(c) If you were asked to give me an example of a connected graph which has no orientations that are strong, what feature(s) would you include in your example to ensure that the example satisfies the requirement? Why??

4. (10 pts.) Theorem 5.17, a corollary of sorts to Menger's Theorem allows you to deal with the vertex connectivity of the graph below easily. Explain briefly. [Hint: Look north-south as well as east-west after considering $\delta(G)$.]



5. (15 pts.) (a) What is a clique?

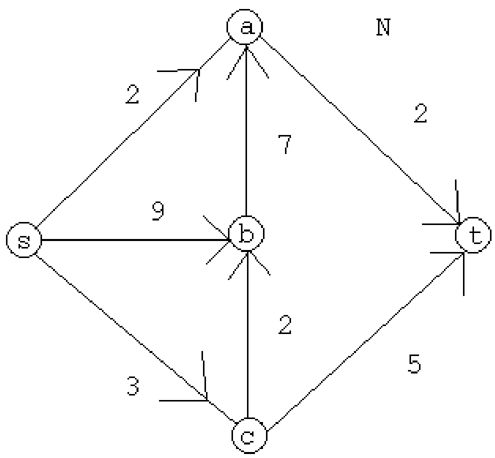
(b) Sketch the shadow graph $S(C_5)$ of a generic 5-cycle below. What is $\chi(S(C_5))$??

(c) How is the Grötzsch graph, which we will denote by G here, obtained from the shadow graph of Part (b) above?? It turns out that $\omega(G) = 2$ and $\chi(G) = 4$. What is the significance of this??

6. (10 pts.) Prove, by induction on the size of the graph, that if G is a connected plane graph of order n, size m, and having r regions, then $n - m + r = 2$.

7. (15 pts.) (a) What is a legal (or feasible) flow in a network N ? [Hint: Definition.]

(b) Obtain a maximum flow f in the network below, and verify the flow is a maximum by producing a set of vertices S that produces a minimum cut. Check that the total capacity of that cut is the same as the value of your max flow.



8. (10 pts.) Which complete bipartite graphs $K_{r,s}$ are Hamiltonian and which are not? Explain briefly. [Hint: When can you use Dirac? What is the well-known necessary condition?]