Student Number: Exam Number:

Carefully use complete sentences and appropriate mathematical notation in answering each of the following questions. Show all essential work; observe that the line of reasoning communicated is especially important. Finally, when displaying your work on a Gauss-Jordan reduction, use the bookkeeping notation established very early in the course; otherwise, no partial credit will be given.

1. (30 pts.) Define each of the following terms or expressions.

(a) Linear combination //

(b) Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ //

(c) Subspace //

(d) Linear independence //

(e) Basis //

(f) Dimension //

(g) Linear Transformation //

(h) Kernel //

(i) Range //

(j) Null Space //

(k) Rank //

(l) Eigenvalue //

(m) Eigenvector //

(n) Invertible (Both senses) //

(o) Orthogonal Complement //

2. (15 pts.) Suppose that B = { $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ } and C = { $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ } are two ordered bases for a vector space V, with $\mathbf{w}_1 = 4\mathbf{v}_1$, $\mathbf{w}_2 = 4\mathbf{v}_1 - 5\mathbf{v}_2$, and $\mathbf{w}_3 = -8\mathbf{v}_1 + 10\mathbf{v}_2 - 2\mathbf{v}_3$.

(a) Compute each of the following.

$[\mathbf{w}_1]_C =$ $[\mathbf{w}_1]_B =$

$[\mathbf{w}_2]_C =$ $[\mathbf{w}_2]_B =$

$[\mathbf{w}_3]_C =$ $[\mathbf{w}_3]_B =$

(b) Obtain the matrix $\underset{B \leftarrow C}{P}$.

(c) What equation does $\underset{B \leftarrow C}{P}$ satisfy for all $\mathbf{x}$ in V ??

(d) Obtain a matrix Q which satisfies $[\mathbf{x}]_C = Q[\mathbf{x}]_B$ for all $\mathbf{x}$ in the vector space V.

3. (20 pts.) Suppose matrices A and B are row equivalent, where

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using this, solve the following problems.

(a) (2 pts.) rank A =

(b) (2 pts.) dim(Nul(A)) =

(c) (3 pt.) Obtain a basis B_1 for Col(A).

B_1 =

(d) (3 pt.) Obtain a basis B_2 for Row(A).

B_2 =

(e) (3 pts.) Obtain a basis B_3 for Nul(A). (It might be wise to work on the back of page 3.)

B_3 =

(f) (4 pts.) Write each column of A which is not a member of B_1 as a linear combination of the elements of B_1.

(g) (3 pts.) Using your work from part (f) of this problem, write an arbitrary linear combination of the columns of A as a linear combination of the elements of B_1.

4. (10 pts.) (a) Starting with the basis

$$B = \{[1\ 1\ 0]^T,\ [1\ 0\ 1]^T,\ [0\ 1\ 1]^T\}$$

for $\mathbb{R}^3$ and working with the usual dot product of $\mathbb{R}^3$, use the Gram-Schmidt process to obtain an orthonormal basis for $\mathbb{R}^3$.

(b) Using your work from part (a), obtain a QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. (10 pts.) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $p_A(\lambda)$ of A. Leave the polynomial in factored form. (Hint: Expand the required determinant using the first row or column. Don't mess with the linear factor which is multiplied by the 2 x 2 determinant, which ends up being a difference of squares.)

$p_A(\lambda)$ =

(b) List the eigenvalues of A.

(c) Obtain a basis, B_λ , for each eigenspace, E_λ , of A. Label correctly.

(d) Obtain an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$.

P = D =

(e) Verify your P and D work without finding the inverse of P. (Hint: There is a matrix equation that is equivalent to the one in part (d).

6. (8 pts.) (a) Show that

$$W = \{a_0 + a_1t + a_2t^2 : a_0 + a_1 + a_2 = 0\}$$

is a subspace of P_2.

(b) Obtain a basis for W.

(Hint: Use the coordinate mapping with respect to the natural basis for P_2 to obtain an equivalent problem in $\mathbb{R}^3$. Solve that problem and translate the results to the polynomial space. Observe that the linear equation you have to solve is the one giving the defining condition on the members of W.)

7. (7 pts.) Let $T: P_2 \to P_2$ be the function defined by the rule

$$T(p) = p'' - 2p',$$

where the primes denote differentiation with respect to t.

(a) Verify that T is a linear transformation. (You need only quote the appropriate properties of differentiation.)

(b) Obtain the matrix $[T]_B$, where $B = \{1, 1 + t, 1 + t + t^2\}$ is an off the wall basis for P_2.

(c) What can you tell about the dimensions of the kernel and the range by using the matrix $[T]_B$??