[Open Book Portion in Class]

Name:

Instructions: **Choose any three of the following problems to solve. Circle the number of each problem you want graded. Given the time constraints, write up your solutions as carefully as possible. Communicate!! If the space allocated for the solution is not sufficient and you continue work in a different location, please indicate at the bottom of the page containing the statement of the problem where that additional work is to be found. You may freely use your textbook and any additional class notes. Please budget your time by allowing approximately 1/3 of the remaining time for each problem you attempt.**

1. Let $f: [0,1] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x^{-1/2}$ for $x > 0$. Prove f is Lebesgue integrable, and compute the value of the Lebesgue integral

$$\int_0^1 f(x)\,dx.$$

Hints: (1) You will need the Monotone Convergence Theorem.
(2) At some point Proposition 4.7 might be useful.
(3) If you want, you may use the Fundamental Theorem of Calculus to evaluate a certain Riemann integral you encounter along the way.

2. Define $f:[0,1] \to \mathbb{R}$ by $f(x) = x$ if x is irrational or $x = 0$, and $f(x) = x + (1/n)$ if $x \neq 0$ is rational and $x = m/n$ in lowest terms.

(a) Give an ε - δ proof that f is not continuous at each rational number $x \in (0,1]$.
(b) Give an ε - δ proof that f is continuous at each irrational number $x \in (0,1]$.
(c) Compute the value of the Riemann integral

$$R \int_0^1 f(x)\, dx.$$

Hints: (1) For each positive integer n there are only finitely many rational numbers with denominators less than n in each finite open interval.
(2) At some point Proposition 4.7 might be useful.
(3) If you want, you may use the Fundamental Theorem of Calculus to evaluate a certain Riemann integral you encounter along the way. Be careful to ensure you have satisfied all the hypotheses, however.

3. Definition: A real-valued function f is **upper semicontinuous at x** if, and only if, for each $\varepsilon > 0$, there's a $\delta > 0$, such that for each t in the domain of f, if $|t - x| < \delta$, then

$$f(t) < f(x) + \varepsilon.$$

Prove Dini's Theorem: Suppose $\langle f_n \rangle$ is a sequence of upper semicontinuous functions defined on a closed and bounded subset A of $\mathbb{R}$ with $f_n(x) \to 0$ as $n \to \infty$ for each $x \in A$, and $f_n(x) \geq f_{n+1}(x) \geq 0$ for $x \in A$ and every n. Then the convergence is uniform on A.

Hint: Let $\varepsilon > 0$. Let $U_n = \{x \in A: f_n(x) < \varepsilon\}$. Show that for each n, $U_n = A \cap O_n$, for some open set $O_n \subset \mathbb{R}$. You must have $U_n \subset U_{n+1}$. [Why?] Use 'closed and bounded' to produce an 'N'. Be very careful with the details.

4. Suppose that $f:[a,b] \to \mathbb{R}$ is continuous and that for some pair of numbers x and y in (a,b) with $x < y$ we have $f(x) > f(y)$. Show that if $D^+f(x) > 0$, then there is a number x_0 in (x,y) with $D^+f(x_0) \leq 0$.

Hint: Define $h(t) = [f(x) - f(t)]/[x - t]$ for $t \in (x,y]$. Find the t where $h(t) = 0$ which is farthest to the right.

5. One point where beginning measure theory students get confused is distinguishing between a measurable function f which is continuous almost everywhere and a function f with the property that there is a continuous function g such that the set $\{x: f(x) \neq g(x)\}$ has measure zero.

(a) Construct an example of a function $f:[0,1] \to \mathbb{R}$ which is continuous nowhere and a function $g:[0,1] \to \mathbb{R}$ which is continuous on all of [0,1], such that $m(\{x: f(x) \neq g(x)\}) = 0$.

(b) Let $\langle q_n \rangle$ be an enumeration of all the rational numbers in the interval [0,1]. Define $f:[0,1] \to \mathbb{R}$ by the following formula:

$$f(x) = \sum_{q_n \le x} 2^{-n}.$$

Here, of course, the sum is over the indices n such that q_n is no larger than x. Note that since the geometric series $\sum 2^{-n}$ dominates the sum defining f, convergence is not a problem.

Show f is continuous at each irrational number in [0,1] and discontinuous at each rational number in [0,1].

Can you find a continuous function $g:[0,1] \to \mathbb{R}$ such that the measure of the set $\{x: f(x) \neq g(x)\}$ is zero? Proof??

Hint: (1) For (a) there are a couple of very easy and obvious functions that do the job.

(2) For (b), it helps to observe that the discontinuities are jumps since f is increasing.

6. (a) Let $f_n(x) = n^{-1} \cdot \chi_{[-n,n]}(x)$ for $n \varepsilon \mathbb{N}$ and $x \varepsilon \mathbb{R}$. Show $\{f_n\}$ converges uniformly to $f(x) = 0$ on $\mathbb{R}$.

(b) With proof determine whether there is a Lebesgue integrable function g defined on $\mathbb{R}$ such the $g(x) \geq f_n(x)$ for every $x \varepsilon \mathbb{R}$ and $n \varepsilon \mathbb{N}$.