MAC 3312

Reviewing Techniques of Integration

1. $\int x \ln x \, dx$
2. $\int \frac{\ln x}{x} dx$
3. $\int \frac{1}{x \ln x} dx$
4. $\int \ln x \, dx$
5. $\int x^2 \sin x \, dx$
6. $\int x \sin(x^2) dx$
7. $\int \sin^2 2x \, dx$
8. $\int \sin^3 2x \, dx$
9. $\int \frac{\sin x}{1+\cos x} dx$
10. $\int \frac{\sin x}{(1+\cos x)^2} dx$
11. $\int \frac{\sin^2 x}{1+\cos x} dx$
12. $\int \sin \sqrt{x} \, dx$
13. $\int \frac{e^x}{1+e^x} dx$
14. $\int \frac{e^x}{1+e^{2x}} dx$
15. $\int x e^{-x^2} dx$
16. $\int x^2 e^{-x} dx$
17. $\int \frac{dx}{x^2+1}$
18. $\int \frac{x \, dx}{x^2+1}$
19. $\int \frac{x^3}{x^2+1} dx$
20. $\int \frac{x+1}{x^2+4} dx$
21. $\int \frac{x+1}{x^2-4} dx$
22. $\int \frac{x+1}{(x-2)^2} dx$
23. $\int \frac{x+1}{x(x-2)^2} dx$
24. $\int \frac{x+2}{x^2(x^2+1)} dx$
25. $\int \frac{dx}{x^2+2x+5}$
26. $\int \frac{x+1}{x^2+2x+5} dx$
27. $\int \frac{dx}{\sqrt{4-x^2}}$
28. $\int \frac{x \, dx}{\sqrt{4-x^2}}$
29. $\int \frac{x^2 dx}{\sqrt{4-x^2}}$
30. $\int \frac{dx}{\sqrt{9-16x^2}}$
31. $\int (\sqrt{x}+2)^2 dx$
32. $\int \frac{(\sqrt{x}+2)^2}{\sqrt{x}} dx$
33. $\int \frac{x}{1+x^4} dx$
34. $\int \frac{dx}{5+3x^2}$
35. $\int x\sqrt{x+3} \, dx$
36. $\int \frac{x+2}{x(x^2+4)} dx$
37. $\int \frac{2x^2+x+4}{(x^2+1)(x^2+4)} dx$
38. $\int \frac{x^3-2x^2-x+1}{x^3(x+1)} dx$
39. $\int \frac{x^2+2}{x^3+x^2-2x} dx$

MAC 3312 **Answers to Review Sheet**

1) $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$u=\ln x \quad dv=x\,dx$

$du=\frac{1}{x}dx \quad v=\frac{1}{2}x^2$

2) $\int \frac{\ln x}{x}dx \quad \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

$u=\ln x$

$du=\frac{1}{x}dx$

3) $\int \frac{1}{x\ln x}dx \quad \int \frac{1}{u}du = \ln|u| = \ln|\ln x| + C$

$u=\ln x$

$du=\frac{1}{x}dx$

4) $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$

$u=\ln x \quad dv=dx$

$du=\frac{1}{x}dx \quad v=x$

5) $\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

$u=x^2 \quad dv=\sin x\,dx$

$du=2x\,dx \quad v=-\cos x$

$u=x \quad dv=\cos x\,dx$

$du=dx \quad v=\sin x$

$= -x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right]$

$= -x^2\cos x + 2x\sin x + 2\cos x + C$

6) $\int x\sin x^2\,dx \quad \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u = -\frac{1}{2}\cos x^2 + C$

$u=x^2$

$du=2x\,dx$

$\frac{1}{2}du = x\,dx$

7) $\int \sin^2 2x\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

8) $\int \sin^3 2x\,dx = \int \sin^2 2x \sin 2x\,dx = \int (1-\cos^2 2x)\sin 2x\,dx$

$u = \cos 2x$

$du = -2\sin 2x\,dx$

$-\frac{1}{2}du = \sin 2x\,dx$

$= -\frac{1}{2}\int (1-u^2)\,du = -\frac{1}{2}u + \frac{1}{6}u^3 = -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + C$

9) $\int \frac{\sin x\,dx}{1+\cos x}$

$u = 1+\cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\rightarrow -\int \frac{du}{u} = -\ln|u| = -\ln|1+\cos x| + C$

10) $\int \frac{\sin x}{(1+\cos x)^2}\,dx$

$u = 1+\cos x$

$-du = \sin x\,dx$

$\rightarrow -\int \frac{1}{u^2}\,du = \frac{-u^{-1}}{-1} = \frac{1}{u} = \frac{1}{1+\cos x} + C$

11) $\int \frac{\sin^2 x}{1+\cos x}\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\,dx = \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}\,dx = \int (1-\cos x)\,dx = x - \sin x + ($

12) $\int \sin\sqrt{x}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$2u\,du = dx$

$\rightarrow \int 2u\sin u\,du$

$U = 2u \quad dV = \sin u\,du$

$dU = 2du \quad V = -\cos u$

$= -2u\cos u + 2\int \cos u\,du$

$= -2u\cos u + 2\sin u + C$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

13) $\int \frac{e^x}{1+e^x}\,dx$

$u = 1+e^x$

$du = e^x dx$

$\rightarrow \int \frac{1}{u}\,du = \ln|u| + C = \ln(1+e^x) + C$

(Absolute value not needed since $1+e^x > 0$ For all x)

14) $\int \frac{e^x}{1+e^{2x}}\,dx$

$u = e^x$

$du = e^x dx$

$\rightarrow \int \frac{du}{1+u^2} = \arctan u + C = \arctan e^x + C$

15) $\int xe^{-x^2}dx$

$u=-x^2$

$du=-2xdx$

$-\frac{1}{2}du=xdx$

$\rightarrow -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$

16) $\int x^2e^{-x}dx$

$u=x^2 \quad dv=e^{-x}dx$

$du=2xdx \quad v=-e^{-x}$

$= -e^{-x}x^2 + 2\int xe^{-x}dx$

$u=x \quad dv=e^{-x}dx$

$du=dx \quad v=-e^{-x}$

$= -x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}dx\right]$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

17) $\int \frac{dx}{x^2+1} = \tan^{-1}x + C$

18) $\int \frac{xdx}{x^2+1}$

$u=x^2+1$

$du=2xdx$

$\frac{1}{2}du=xdx$

$\rightarrow \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$

19) $\int \frac{x^3dx}{x^2+1} = \int \left(x - \frac{x}{x^2+1}\right)dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$

$x^2+1 \overline{)\, x^3}$ $\quad x$

x^3+x

$-x$

(2nd term of answer comes from problem #18)

20) $\int \frac{x+1}{x^2+4}dx = \int \frac{x}{x^2+4}dx + \int \frac{1}{x^2+4}dx$

$u=x^2+4$

$du=2xdx$

$\frac{1}{2}du=xdx$

$= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C$

21) $\int \frac{x+1}{x^2-4}\,dx = \int \frac{x+1}{(x-2)(x+2)}\,dx$

$$\frac{x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$x+1 = A(x+2)+B(x-2)$$

$x=2 \Rightarrow 3=4A \Rightarrow A=\frac{3}{4}$

$x=-2 \Rightarrow -1=-4B \Rightarrow B=\frac{1}{4}$

$$= \frac{3}{4}\int \frac{1}{x-2}\,dx + \frac{1}{4}\int \frac{1}{x+2}\,dx = \frac{3}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C$$

22) $\int \frac{x+1}{(x-2)^2}\,dx$

$$\frac{x+1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$x+1 = A(x-2)+B$$

$x=2 \Rightarrow 3=B$

$x=1 \Rightarrow 2=-A+B \Rightarrow 2=-A+3 \Rightarrow A=1$

$$\int \frac{1}{x-2}\,dx + 3\int \frac{1}{(x-2)^2}\,dx = \ln|x-2| - \frac{3}{x-2} + C$$

ALTERNATE METHOD

$\int \frac{x+1}{(x-2)^2}\,dx$

$u=x-2$

$du=dx$

$u+3=x+1$

$$\int \frac{u+3}{u^2}\,du = \int \frac{u}{u^2}\,du + \int \frac{3}{u^2}\,du = \int \frac{1}{u}\,du + 3\int \frac{1}{u^2}\,du$$

$$= \ln|u| - \frac{3}{u} + C$$

$$= \ln|x-2| - \frac{3}{x-2} + C$$

23) $\int \frac{x+1}{x(x-2)^2}\,dx$

$$\frac{x+1}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x+1 = A(x-2)^2 + Bx(x-2) + Cx$$

$x=0 \Rightarrow 1=4A \Rightarrow A=\frac{1}{4}$

$x=2 \Rightarrow 3=2C \Rightarrow C=\frac{3}{2}$

$x=1 \Rightarrow 2=A-B+C \Rightarrow 2=\frac{1}{4}-B+\frac{3}{2} \Rightarrow B=-\frac{1}{4}$

(continued next page)

23(cont.) $\frac{1}{4}\int\frac{1}{x}dx - \frac{1}{4}\int\frac{1}{x-2}dx + \frac{3}{2}\int\frac{1}{(x-2)^2}dx$

$\frac{1}{4}\ln|x| - \frac{1}{4}\ln|x-2| - \frac{3}{2(x-2)} + C$

24) $\int\frac{x+2}{x^2(x^2+1)}dx$

$\frac{x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$x+2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

$x+2 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$

$x+2 = (A+C)x^3 + (B+D)x^2 + Ax + B$

$A+C=0$

$B+D=0$

$A=1 \qquad C=-1$

$B=2 \qquad D=-2$

$\int\frac{1}{x}dx + 2\int\frac{1}{x^2}dx + \int\frac{-x-2}{x^2+1}dx$

$\int\frac{1}{x}dx + 2\int\frac{1}{x^2}dx - \int\frac{x\,dx}{x^2+1} - 2\int\frac{1}{x^2+1}dx$

$\ln|x| - \frac{2}{x} - \frac{1}{2}\ln(x^2+1) - 2\tan^{-1}x + C$

25) $\int\frac{dx}{x^2+2x+5} = \int\frac{dx}{(x^2+2x+1)+4} = \int\frac{dx}{(x+1)^2+4} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$

26) $\int\frac{x+1}{x^2+2x+5}dx$

$u = x^2+2x+5$

$du = (2x+2)dx$

$\frac{1}{2}du = (x+1)dx$

$\rightarrow \frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x+5| + C$

27) $\int\frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + C$

28) $\int\frac{x\,dx}{\sqrt{4-x^2}}$

$u = 4-x^2$

$du = -2x\,dx$

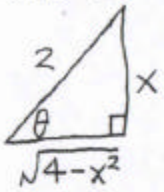
$-\frac{1}{2}du = x\,dx$

$\rightarrow -\frac{1}{2}\int\frac{1}{\sqrt{u}}du = -\frac{1}{2}\int u^{-\frac{1}{2}}du = -u^{\frac{1}{2}} + C = -\sqrt{4-x^2} + C$

29) $\int \frac{x^2\,dx}{\sqrt{4-x^2}}$

$x = 2\sin\theta$ or $\theta = \sin^{-1}\frac{x}{2}$

$dx = 2\cos\theta\,d\theta$



$$\int \frac{4\sin^2\theta(2\cos\theta)\,d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{4\sin^2\theta(2\cos\theta)\,d\theta}{\sqrt{4(1-\sin^2\theta)}}$$

$$= \int \frac{4\sin^2\theta(2\cos\theta)\,d\theta}{\sqrt{4\cos^2\theta}} = \int \frac{4\sin^2\theta(2\cos\theta)\,d\theta}{2\cos\theta}$$

$$= \int 4\sin^2\theta\,d\theta = 4\int\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$= 4\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + C = 2\theta - \sin 2\theta + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C = 2\sin^{-1}\frac{x}{2} - 2\left(\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2\sin^{-1}\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C$$

30) $\int \frac{dx}{\sqrt{9-16x^2}} = \int \frac{dx}{\sqrt{16\left(\frac{9}{16}-x^2\right)}} = \int \frac{dx}{4\sqrt{\frac{9}{16}-x^2}} = \frac{1}{4}\int \frac{dx}{\sqrt{\left(\frac{3}{4}\right)^2-x^2}} = \frac{1}{4}\sin^{-1}\left(\frac{x}{3/4}\right)+C$

$$= \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + C$$

31) $\int (\sqrt{x}+2)^2\,dx = \int (x + 4\sqrt{x} + 4)\,dx = \frac{1}{2}x^2 + \frac{8}{3}x^{3/2} + 4x + C$

32) $\int \frac{(\sqrt{x}+2)^2}{\sqrt{x}}$

$u = \sqrt{x}+2$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2du = \frac{dx}{\sqrt{x}}$

$$2\int u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(\sqrt{x}+2)^3 + C$$

33) $\int \frac{x}{1+x^4}\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}x^2 + C$$

34) $\int \frac{dx}{5+3x^2} = \int \frac{dx}{3\left(\frac{5}{3}+x^2\right)} = \frac{1}{3}\frac{1}{\sqrt{5/3}}\arctan\frac{x}{\sqrt{5/3}} + C = \frac{1}{3}\frac{\sqrt{3}}{\sqrt{5}}\arctan\sqrt{\frac{3}{5}}x + C$

$$= \frac{\sqrt{15}}{15}\arctan\frac{\sqrt{15}x}{5} + C$$

35) $\int x\sqrt{x+3}\,dx$

$u = x+3$

$du = dx$

$u-3 = x$

$\int (u-3)\sqrt{u}\,du = \int (u^{3/2} - 3u^{1/2})\,du = \frac{2}{5}u^{5/2} - 2u^{3/2} + C$

$= \frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C$

36) $\int \frac{x+2}{x(x^2+4)}\,dx$

$\frac{x+2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$x+2 = A(x^2+4) + (Bx+C)x$

$x=0 \Rightarrow 2 = 4A \Rightarrow A = \frac{1}{2}$

Multiplying out: $x+2 = Ax^2 + 4A + Bx^2 + Cx$

$x+2 = (A+B)x^2 + Cx + 4A$

$A+B=0$

$C=1$ $\quad \searrow B = -\frac{1}{2}$

$4A=2$

$\frac{1}{2}\int \frac{1}{x}\,dx + \int \frac{-\frac{1}{2}x+1}{x^2+4}\,dx$

$\frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{x\,dx}{x^2+4} + \int \frac{dx}{x^2+4}$

$\frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

37) $\int \frac{2x^2+x+4}{(x^2+1)(x^2+4)}\,dx$

$\frac{2x^2+x+4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$

$2x^2+x+4 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$

$2x^2+x+4 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$

$2x^2+x+4 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)$

$A+C=0$

$B+D=2$

$4A+C=1$

$4B+D=4$

$\Rightarrow A = \frac{1}{3} \quad B = \frac{2}{3} \quad C = -\frac{1}{3} \quad D = \frac{4}{3}$

$\int \frac{\frac{1}{3}x+\frac{2}{3}}{x^2+1}\,dx + \int \frac{-\frac{1}{3}x+\frac{4}{3}}{x^2+4}\,dx = \frac{1}{3}\int \frac{x}{x^2+1}\,dx + \frac{2}{3}\int \frac{dx}{x^2+1} - \frac{1}{3}\int \frac{x\,dx}{x^2+4} + \frac{4}{3}\int \frac{dx}{x^2+4}$

$= \frac{1}{6}\ln(x^2+1) + \frac{2}{3}\arctan x - \frac{1}{6}\ln(x^2+4) + \frac{2}{3}\arctan\frac{x}{2} + C$

38) $\int \frac{x^3-2x^2-x+1}{x^3(x+1)}\,dx$

$$\frac{x^3-2x^2-x+1}{x^3(x+1)} = \frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{x+1}$$

$$x^3-2x^2-x+1 = A(x+1)+Bx(x+1)+Cx^2(x+1)+Dx^3$$

$x=0 \Rightarrow 1=A$

$x=-1 \Rightarrow -1=-D \Rightarrow D=1$

Multiplying out: $x^3-2x^2-x+1 = Ax+A+Bx^2+Bx+Cx^3+Cx^2+Dx^3$

$$x^3-2x^2-x+1 = (C+D)x^3+(B+C)x^2+(A+B)x+A$$

$C+D=1 \Rightarrow C+1=1 \Rightarrow C=0$

$B+C=-2 \Rightarrow B+0=-2 \Rightarrow B=-2$

$$\int \frac{1}{x^3}dx - 2\int \frac{1}{x^2}dx + \int \frac{dx}{x+1} = \frac{-1}{2x^2}+\frac{2}{x}+\ln|x+1|+C$$

39) $\int \frac{x^2+2}{x^3+x^2-2x}\,dx$

$$\frac{x^2+2}{x(x+2)(x-1)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-1}$$

$$x^2+2 = A(x+2)(x-1)+Bx(x-1)+Cx(x+2)$$

$x=0 \Rightarrow 2=-2A \Rightarrow A=-1$

$x=1 \Rightarrow 3=3C \Rightarrow C=1$

$x=-2 \Rightarrow 6=6B \Rightarrow B=1$

$$\int \frac{-1}{x}dx + \int \frac{dx}{x+2} + \int \frac{dx}{x-1}$$

$$-\ln|x|+\ln|x+2|+\ln|x-1|+C$$