MAC 2312 **Review of sections 10.3 - 10.6**

For problems 1-14, determine whether the series converges or diverges. Justify your answer.

1) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

2) $\sum_{k=1}^{\infty} \frac{1}{2^k}$

3) $\sum_{k=1}^{\infty} \frac{1}{2k}$

4) $\sum_{k=1}^{\infty} \frac{1}{(2k)!}$

5) $\sum_{k=5}^{\infty} \frac{4}{k^2 - 4}$

6) $\sum_{k=5}^{\infty} \frac{4}{(k-4)^2}$

7) $\sum_{k=5}^{\infty} \frac{4}{k^2 + 4}$

8) $\sum_{k=0}^{\infty} e^{-k}$

9) $\sum_{k=1}^{\infty} e^{-\frac{1}{k}}$

10) $\sum_{k=1}^{\infty} ke^{-k^2}$

11) $\sum_{k=2}^{\infty} \frac{k}{\ln k}$

12) $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

13) $\sum_{k=3}^{\infty} \frac{\sqrt{k}}{k^2 + 7}$

14) $\sum_{k=3}^{\infty} \frac{\sqrt{k}}{k^2 - 7}$

15) For which of the convergent series above can you find the sum? Go ahead and find those sums.

For problems 16-19, determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

16) $\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)^{10}}$

17) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k - 1}$

18) $\sum_{k=0}^{\infty} \frac{(-1)^k (k+1)!}{3^k}$

19) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

SOLUTIONS

1) Converges because it is a p-series with p = 2. Or, if you prefer doing things the hard way, you can use the integral test.

2) Converges because it is geometric with r = ½ . Hard way: ratio test or integral test.

3) Diverges because it is the harmonic series times a constant. (Or a p-series with p = 1 times a constant.) Hard way: integral test.

4) $$\frac{u_{k+1}}{u_k} = \frac{1}{[2(k+1)]!}\cdot\frac{(2k)!}{1} = \frac{(2k)!}{(2k+2)(2k+1)(2k)!} = \frac{1}{(2k+2)(2k+1)}$$

$$\lim_{k\to\infty}\frac{1}{(2k+2)(2k+1)} = 0 < 1$$

Converges by the ratio test.

5) This series is telescoping

$$\frac{4}{(k+2)(k-2)} = \frac{A}{k+2} + \frac{B}{k-2}$$

$$4 = A(k-2) + B(k+2)$$

$$k = 2 \Rightarrow 4 = 4B \Rightarrow B = 1$$

$$k = -2 \Rightarrow 4 = -4A \Rightarrow A = -1$$

$$\sum_{k=5}^{\infty}\left(\frac{1}{k-2} - \frac{1}{k+2}\right)$$

$$s_n = \left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{6}-\frac{1}{10}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n+2}\right)+$$
$$\left(\frac{1}{n-1}-\frac{1}{n+3}\right)+\left(\frac{1}{n}-\frac{1}{n+4}\right)+\left(\frac{1}{n+1}-\frac{1}{n+5}\right)+\left(\frac{1}{n+2}-\frac{1}{n+6}\right)$$

$$s_n = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{n+3}-\frac{1}{n+4}-\frac{1}{n+5}-\frac{1}{n+6}$$

$$\lim_{n\to\infty} s_n = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \frac{57}{60}$$

Converges (You could have also done the integral test here, using partial fractions or the substitution x = 2sec2 to antidifferentiate.)

6) Converges because it is a p-series with p=2 times a constant.

7)
$$\int_5^\infty \frac{4dx}{x^2+4} = \lim_{b\to\infty}\int_5^b \frac{4dx}{x^2+4} = \lim_{b\to\infty}\left[4\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_5^b = \lim_{b\to\infty}\left[2\tan^{-1}\frac{b}{2} - 2\tan^{-1}\frac{5}{2}\right]$$
$$= 2\left(\frac{\pi}{2}\right) - 2\tan^{-1}\frac{5}{2}$$

Converges by the integral test (Or you could have used the comparison test comparing it to $\sum_{k=5}^{\infty}\frac{4}{k^2}$ which is a p-series with p = 2, times a constant, with the first 4 terms deleted.)

8) Converges because it is geometric with r = 1/e. Hard way: ratio test or integral test.

9) $\lim_{k\to\infty} e^{-\frac{1}{k}} = e^0 = 1 \neq 0$ Diverges by the divergence test.

10)
$$\frac{u_{k+1}}{u_k} = \frac{k+1}{e^{(k+1)^2}}\cdot\frac{e^{k^2}}{k} = \frac{(k+1)e^{k^2}}{ke^{k^2+2k+1}} = \frac{(k+1)e^{k^2}}{ke^{k^2}\cdot e^{2k+1}} = \frac{k+1}{ke^{2k+1}}$$
$$\lim_{k\to\infty}\frac{k+1}{ke^{2k+1}} = 0 < 1$$

Converges by the ratio test. (You could also have done the integral test here using the substitution $u = -x^2$.

11)
$$\lim_{k\to\infty}\frac{k}{\ln k} = \lim_{x\to\infty}\frac{x}{\ln x} = \lim_{x\to\infty}\frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty \neq 0$$

Diverges by the divergence test.

12)
$$f(x) = \frac{\ln x}{x}$$
$$f'(x) = \frac{x\cdot\frac{1}{x} - \ln x\cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0 \quad \forall x \geq 3$$

So *f* is decreasing on [3, 4). *f* is also continuous on the interval and has all positive terms.

$$\int_3^\infty \frac{\ln x}{x}dx = \lim_{b\to\infty}\int_3^b \frac{\ln x}{x}dx = \lim_{b\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_3^b = \lim_{b\to\infty}\left[\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 3)^2\right] = +\infty$$

Diverges by the integral test.

13) $\dfrac{\sqrt{k}}{k^2+7} < \dfrac{\sqrt{k}}{k^2} = \dfrac{1}{k^{\frac{3}{2}}}$

$\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges because it is a p-series with p = 1.5, with the first 2 terms deleted.

$\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k}}{k^2+7}$ converges by the comparison test.

14) $\dfrac{\sqrt{k}}{k^2-7} < \dfrac{\sqrt{k}}{k^2 - \frac{1}{2}k^2} = \dfrac{\sqrt{k}}{\frac{1}{2}k^2} = \dfrac{2}{k^{\frac{3}{2}}} \quad \forall k \geq 4$

$\displaystyle\sum_{k=3}^{\infty} \frac{2}{k^{\frac{3}{2}}}$ converges because it is a p-series with p = 1.5, times a constant, with the first 2 terms deleted.

$\displaystyle\sum_{k=3}^{\infty} \frac{\sqrt{k}}{k^2-7}$ converges by the comparison test.

15) #2 has a sum of $\dfrac{a}{1-r} = \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

#5 has a sum of $\dfrac{57}{60}$

#8 has a sum of $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{e}} = \dfrac{e}{e-1}$

16) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+1)^{10}}$ converges because it is a p-series with p = 10, with the first term deleted.

So $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)^{10}}$ is absolutely convergent.

17) $\lim_{k\to\infty} \frac{(-1)^{k+1}}{3k-1} = 0$ and $\frac{1}{2} > \frac{1}{5} > \frac{1}{8} > \ldots$ so it converges by the alternating series test.

$$\int_1^\infty \frac{dx}{3x-1} = \lim_{b\to\infty} \int_1^b \frac{dx}{3x-1} = \lim_{b\to\infty} \left[\frac{1}{3} \ln|3x-1| \right]_1^b = \frac{1}{3} \lim_{b\to\infty} \left[\ln|3b-1| - \ln 2 \right] = +\infty$$

So $\sum_{k=1}^{\infty} \frac{1}{3k-1}$ diverges by the integral test.

Thus $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-1}$ is conditionally convergent.

18) $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k\to\infty} \frac{(k+2)!}{3^{k+1}} \cdot \frac{3^k}{(k+1)!} = \lim_{k\to\infty} \frac{(k+2)(k+1)! \cdot 3^k}{(k+1)!\, 3^k \cdot 3^1} = \lim_{k\to\infty} \frac{k+2}{3} = +\infty$

Diverges by the ratio test for absolute convergence.

19) $\frac{\cos 1}{1} + \frac{\cos 2}{4} + \frac{\cos 3}{9} + \frac{\cos 4}{16} + \frac{\cos 5}{25} + \ldots$

Using a calculator (in the radian mode), you can confirm that the first term is positive, the next 3 terms are negative, etc. In other words, this is not an alternating series. We will consider the series of absolute values: $\sum_{k=1}^{\infty} \frac{|\cos k|}{k^2}$

Since $|\cos k| \le 1 \quad \forall k$, we have $\frac{|\cos k|}{k^2} \le \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges because it is a p-series with p = 2.

$\sum_{k=1}^{\infty} \frac{|\cos k|}{k^2}$ converges by the comparison test

$\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ is absolutely convergent.