MAP 2302 Section 6.2

What happens when the exponents of the D.E. are imaginary?

$x^2y''+xy'+y=0$ (This can be solved as a Cauchy-Euler equation, but lets use the method of Frobenius) First, note that 0 is a regular singular point.

Assume a solution of the form $y=\sum_{n=0}^{\infty} C_n x^{n+r}$, $C_0\neq 0$, $x>0$.

Substituting y, y', and y'' in the D.E. yields:

$$\sum_{n=0}^{\infty}(n+r-1)(n+r)C_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)C_n x^{n+r}+\sum_{n=0}^{\infty}C_n x^n=0$$

Setting $n=0$ to get the indicial equation $\rightarrow$ $(r-1)r+r+1=0$

$$r^2+1=0$$
$$r=\pm i$$

For $n\geq 1$ $(n+r-1)(n+r)C_n+(n+r)C_n+C_n=0$

$$(n+r)[n+r-1+1]C_n+C_n=0$$
$$[(n+r)^2+1]C_n=0$$

Case I: $r=i$

$[(n+i)^2+1]C_n=0$

$[n^2+2in]C_n=0$

$n(n+2i)C_n=0$

Since $n\geq 1$, $n\neq 0$ and $n\neq -2i$. So $C_n=0$

Thus all C_i except C_0 are zero.

We get the solution $y_1=C_0x^i$

Case II: $r=-i$

$[(n-i)^2+1]C_n=0$

$[n^2-2in]C_n=0$

$n(n-2i)C_n=0$

Since $n\geq 1$, $n\neq 0$ and $n\neq 2i$. So $C_n=0$

Thus all C_i except C_0 are zero.

We get the solution $y_2=C_0^* x^{-i}$

Using the identity $e^{ix}=\cos x+i\sin x$:

$$x^i=(e^{\ln x})^i=e^{i\ln x}=\cos(\ln x)+i\sin(\ln x)$$

$$x^{-i}=(e^{\ln x})^{-i}=e^{-i\ln x}=\cos(-\ln x)+i\sin(-\ln x)=\cos(\ln x)-i\sin(\ln x)$$

$x^i+x^{-i}=2\cos(\ln x)\Rightarrow \cos(\ln x)=\frac{1}{2}x^i+\frac{1}{2}x^{-i}\Rightarrow \cos(\ln x)$ is a linear combination of solutions $\Rightarrow \cos(\ln x)$ is a solution.

$x^i-x^{-i}=2i\sin(\ln x)\Rightarrow \sin(\ln x)=\frac{1}{2i}x^i-\frac{1}{2i}x^{-i}\Rightarrow \sin(\ln x)$ is a linear combination of solutions $\Rightarrow \sin(\ln x)$ is a solution.

Since $\cos(\ln x)$ and $\sin(\ln x)$ are linearly independent (verify using Wronskian) on $(0,+\infty)$, the general solution can be written $y=C_1\cos(\ln x)+C_2\sin(\ln x)$