MGF 1107 PROBLEM SET 12

Our final chapter is about Game Theory, the mathematical study of strategy. For our first game, let's consider a very simplified version of a battle between two armies: the Yankees and the Rebels. The Yankees have two possible strategies, attack by land or attack by air. The Rebels also have two strategies, defend against a possible land attack or defend against a possible air attack. There are four possible outcomes in this battle:
i) The Yankees attack by air and the Rebels defend against an air attack.
ii) The Yankees attack by air and the Rebels defend against a land attack.
iii) The Yankees attack by land and the Rebels defend against an air attack.
iv) The Yankees attack by land and the Rebels defend against a land attack.
Generals agree that in the first case the Yankees kill 200 Rebels. We can display this information as follows.

	Rebels defend against an air attack	Rebels defend against a land attack
Yankees attack by air	(200, -200)	
Yankees attack by land		

The first number is the *payoff* to the Yankees, a gain of 200 for killing 200 of the Rebels.
The second number is the payoff to the Rebels. It is negative because they lost 200 men.
The generals assign the following probable numbers to the other three possible outcomes:
ii) If the Yankees attack by air and the Rebels defend against a land attack, the Yankees kill 400 rebels.
iii) If the Yankees attack by land and the Rebels defend against an air attack, the Yankees kill 150 rebels.
iv) If the Yankees attack by land and the Rebels defend against a land attack, the Yankees kill 100 rebels.
1. Use this information to complete the table above, which is called a *payoff matrix*. A matrix is the name mathematicians give to what computer users call a spreadsheet.

In a matrix rows extend in the left-right direction and columns extend in the up down direction. In the game above, we call the Yankees the *row player* and we call the Rebels the *column player*. Row 1 is the row that tells us the payoffs if the Yankees attack by air and row 2 is the row that tells us the payoffs if the Yankees attack by land. Column 1 is the column that tells us the payoffs if the Rebels defend against an air attack and column 2 is the column that tells us the payoffs if the Rebels defend against a land attack.

2. Find the sum of each pair of numbers in the matrix.

For the reason you saw in problem 2, games such as the one above are called *zero-sum games*. In a zero sum game, once you know the payoff to the row player you automatically know the payoff to the column player (it is the negative of the playoff to the row player). For that reason, in a zero-sum game we don't need to list the payoffs to the column player. The matrix above is typically written like this:

	Rebels defend against an air attack	Rebels defend against a land attack
Yankees attack by air	200	400
Yankees attack by land	150	100

3. Let's consider a different zero-sum game for a moment. We will call the row players two strategies R1 and R2 and the column player's two strategies C1 and C2.

	C1	C2
R1	5	6
R2	-2	1

a) What is the payoff to the row player if row uses strategy R1 and the column players uses strategy C1?
b) What is the payoff to the column player if row uses strategy R1 and the column players uses strategy C1?
c) What is the payoff to the column player if row uses strategy R2 and the column players uses strategy C1?
d) If you were the row player in this game which strategy would you choose and why?

e) If you were the column player in this game which strategy would you choose and why?

A strategy that is always better than another strategy is called a *dominant strategy*. In problem 3, the row player's dominant strategy was R1 and the column player's dominant strategy was C1.

4. Returning to the Yankees-Rebels war game, does either play have a dominant strategy?

5. If you were the commander of the Rebels, which strategy would you choose and why?

6. In a zero-sum game where we write only the payoffs to the row player,
a) does the row player prefer big numbers or small numbers?

b) does the column player prefer big numbers or small numbers?

7. There is another way to analyze the Yankees-Rebels war game. The fundamental idea in Game Theory is to look at the worst case scenario presented by each option, then choose the best of the worst case scenarios.
a) What is the worst possible payoff for the Yankees if they attack by air?

b) What is the worst possible payoff for the Yankees if they attack by land?

These numbers are called the *row minima*. Minima is the plural form of the word minimum.
c) What is the best of the worst case scenarios, i.e. the maximum of the row minima?

This number is called the *maximin*.
d) What is the worst possible payoff for the Rebels if they defend against an attack by air?

e) What is the worst possible payoff for the Rebels if they defend against an attack by land?

These numbers are called the column *maxima*. Maxima is the plural form of the word maximum.
f) What is the best of the worst case scenarios, i.e. the minimum of the row maxima?

This number is called the *minimax*. When the maximin is the same number as the minimax, we call that number the *saddle point*. Not all zero-sum games have saddle points, but when a saddle point exists, the strategies that produce that payoff are the best strategy for each player to follow.

For the following games, the row player has 3 possible strategies R1, R2 and R3, and the column player has 3 possible strategies C1, C2 and C3.
a) Find the maximin
b) Find the minimax
c) Determine if the game has a saddle point
d) For the games that have a saddle point, find each player's best strategy.
8.

	C1	C2	C3
R1	3	4	0
R2	5	6	2
R3	4	1	1

9.

	C1	C2	C3
R1	3	4	-1
R2	-5	6	2
R3	4	1	3