MGF 1107 PROBLEM SET 13

Let’s oversimplify the game of American football so as to assume the offense has only two strategies (run or pass) and the defense has only two strategies (defend against the run and defend against the pass).

1. Consider a “game” to consist of one play (a pass or run). Is this “game” a zero-sum game?

2. The Panthers are playing the Owls. Statistics from past games tell the coaches the following:

- If the Panthers run and the Owls defend against the run, the Panthers will gain 1 yard.
- If the Panthers run and the Owls defend against the pass, the Panthers will gain 5 yards.
- If the Panthers pass and the Owls defend against the run, the Panthers will gain 6 yards.
- If the Panthers pass and the Owls defend against the pass, the Panthers will gain 2 yards.

Making the Panthers the row player, find the payoff matrix for this game.

$$\begin{array}{ccc} & & \text{Owls defend against the} \\ & & \begin{array}{cc} \text{Run} & \text{Pass} \end{array} \\ \text{Panthers} & \begin{array}{c} \text{Run} \\ \text{Pass} \end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

A game, like the one above, where both “players” have two strategies, is called a 2-by-2 game.

3. Find the maximin and the minimax and check to see if this game has a saddle point.

It should be obvious that the optimal strategy for both teams is to mix up their strategies. For example, if the Panthers were to always run, the Owls would know to always defend against the run. The optimal strategy for a game is always one of two possible types:

i) a *pure strategy*, meaning the player should choose the same strategy every time the game is played

ii) a *mixed strategy*, meaning the player should sometimes use one strategy and sometimes another strategy.

When a game has a saddle point, the optimal strategy will be a pure strategy and when a game does not have a saddle point, the optimal strategy will be a mixed strategy. There are many ways the Panthers can mix their strategies. Some examples are:

- run ½ of the time and pass ½ of the time
- run ¼ of the time and pass ¾ of the time
- run ¾ of the time and pass ¼ of the time

Our goal is to find the optimal mixed strategy for *both* teams.

4. The two fractions in the optimal mixed strategy must always add up to what number?

We will let p denote the proportion of the time the Panthers run and let 1 – p denote the proportion of the time the Panthers pass. Let’s see how many yards the Panthers can expect to gain for each of the Owls’ two strategies. Recall, to find an expected value, we multiply each outcome by its probability (i.e. proportion), then add.

The expected gain when the Owls defend against the run, which can be abbreviated E(Owls defend against run), is obtained by multiplying each outcome that can occur when the Owls defend against the run by the proportion of the time that outcome occurs:
E(Owls defend against run) = $1p + 6(1 – p)$
Removing the parentheses and combining like terms gives us:
E(Owls defend against run) = $1p + 6(1 – p) = 1p + 6 – 6p = -5p + 6$

5. Find the expected gain when the Owls defend against the pass.

If we let y = the expected number of yards gained, we get two linear equations:

$y = -5p + 6$ and $y = 3p + 2$

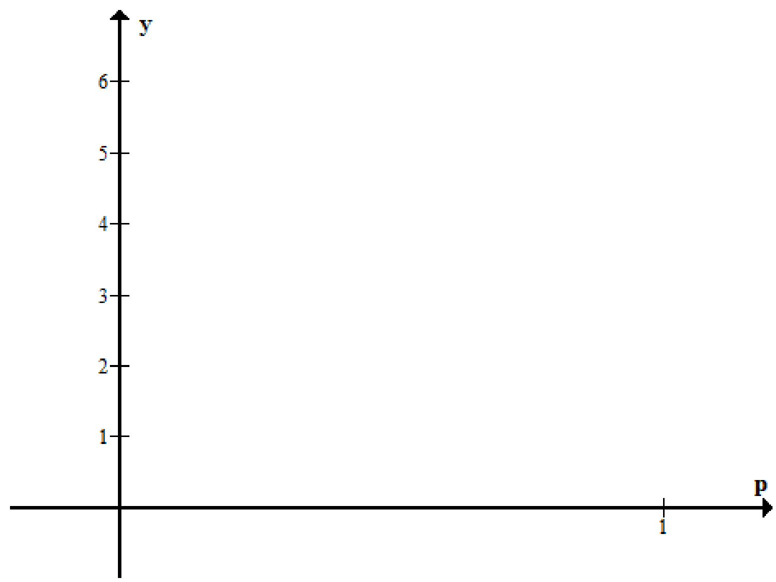
We are going to graph these two lines. Two points are needed to graph a line and, since p is a probability, it must be between 0 and 1.

6. For each line, plug in 0 for p and then 1 for p to get the corresponding y-coordinates.

$y = -5p + 6$ and $y = 3p + 2$

p	y
0	
1	

p	y
0	
1	



7. Plot each point on the given axes and connect them with a straightedge (e.g. the side of your student ID) to get the two lines.

8. The fundamental idea in Game Theory is to maximize your worst-case scenario. The worst-case scenario to the Panthers is the minimum payoff (i.e. the smallest number of yards gained). The minimum payoff is given by that part of the graph that lies lowest (i.e. closest to the p-axis). It is the part of the line $3p + 2$ to the left of the intersection point and the part of the line $– 5p + 6$ to the right of the intersection point. What point maximizes the worst-case scenario?

9. Algebraically find the coordinates of this point by setting the two equations equal to each other and solve for p.

So the optimal strategy for the Panthers is to run the ball ½ of the time.

10. What proportion of the time should the Panthers throw the ball?

It is important that the Panthers not execute the strategy as follows:

Run, pass, run, pass, run, pass,….

because that pattern is predictable. They need to run ½ the time and pass ½ the time in an unpredictable manner. One way to achieve this is by tossing a coin on each play and running if it lands on heads and passing if it lands on tails (but try convincing the Panthers coach that this is the best way to call plays!).

By the way, the optimal strategy is ALWAYS the point of intersection so, in the future, we can skip the graph and just set the two expected values equal to one another. The reason for drawing the graph here is to see ***why*** we equate the two expected values.

Now we are going to find the Owls' optimal mixed strategy.
Let q = the proportion of the time the Owls defend against the run
and 1 – q = the proportion of the time the Owls defend against the pass.
Again, we are going to find two expected values, the expected values corresponding to each of the opponent's (i.e. the Panthers) two strategies.
E(Panthers run) = $1q + 5(1 - q) = 1q + 5 - 5q = -4q + 5$

11. Find the expected value when the Panthers pass the ball.
12. Set these two expected values equal to each other and solve for q.

So the optimal strategy for the Owls' defense is to defend against the run $\frac{3}{8}$ of the time.

13. What proportion of the time should the Owls defend against the pass?

Since we have found both players' optimal strategy, we have *solved the game*. Now let's suppose both players follow their optimal strategies for one play. We want to find the expected gain for the Panthers.
14. What is the probability that the Panthers run the ball and the Owls defend against the run?

15. Complete the following table by, where the first row lists all the possible yards gained and the second row contains the corresponding probabilities.

Yards gained	1	5	6	2
Probability	$\frac{3}{16}$			

16. Find the expected gain per play by multiplying each value by its probability, then adding.

This means that if both players follow their optimal strategies, and every time we played this game, we kept a list of the yards gained by the Panthers, after a large number of plays, the average gain per play would be 3.5 yards. This number is called the *value of the game*.