MGF 1107 PROBLEM SET 5

1. A county consists of 3 cities, Anastasia, Blue Lakes, and Condo Canyon. The county commission consists of 3 commissioners, one from each city. The number of votes each commissioner gets depends on the population of the city. The current census gives the commissioner from Anastasia 3 votes, the commissioner from Blue Lakes 2 votes and the commissioner from Condo Canyon 1 vote. A majority of votes is needed to win so we can describe this weighted voting system as [4: 3, 2, 1]. We are going to consider a different way of measuring power from the Banzhaf method. Suppose that the commissioners join a coalition one at a time (after they have been convinced to vote that way) and we consider the order in which they join the coalition. There are 6 orders in which commissioners can join the coalition:

ABC (which means A joins first, followed by B, and lastly C)
ACB (which means A joins first, followed by C, and lastly B)
BAC
BCA
CAB
CBA

The voter who has the power in each coalition is the voter who, when he joins the coalition, causes it to go from losing to winning. For example, in the coalition ABC, when A joins it gives the coalition 3 votes, which is not enough to win. When B joins, the coalition now has $3 + 2 = 5$ votes, which is enough to win. Since B is the voter who, when he joined the coalition, caused it to go from losing to winning, B is called the *pivotal voter*.

a) Complete the table by finding the pivotal voter in the other 5 coalitions.

Coalition	Pivotal voter
ABC	B
ACB	
BAC	
BCA	
CAB	
CBA	

b) Notice that A is pivotal in 4 of the 6 coalitions. We use the fraction $\frac{4}{6}$ or $\frac{2}{3}$ to describe A's power. What fractions describe the power of voters B and C?

We can write this information as $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$. We call this the *Shapley-Shubik power index* of the weighted voting system. Here are the important differences between the Banzhaf power index and the Shapley-Shubik power index.

- Shapley-Shubik coalitions contain every voter whereas Banzhaf coalitions can contain any number of voters.
- In a Shapley-Shubik coalition, order matters. In other words, ABC and CBA are different coalitions. Banzhaf considers them to be the same coalition.
- Every Shapley-Shubik coalition has exactly one pivotal voter; no more, no less. Banzhaf coalitions can have more than one critical voter or no critical voter at all.
- Shapley-Shubik describes the power of each voter with fractions while Banzhaf uses whole numbers.
- The fractions in a Shapley-Shubik power index must always add up to 1.

2. How could you have figured out that there were 6 coalitions without listing them all?
Hint: Use combinatorics.

3. How many Shapley-Shubik coalitions are there if we have:
a) 4 voters?

b) 5 voters?

4. A law firm has seven partners: a senior partner with 6 votes and six junior partners with 1 vote each. The quota is a majority of the votes.
a) Describe this weighted voting system in the form [q: w1, w2, w3, w4, w5, w6, w7].

b) How many Shapley-Shubik coalitions does this weighted voting system have?

c) Since there are too many coalitions to list them all, we will use a shortcut to find the Shapley-Shubik power index. First, we will find the power index of the senior partner, who we will call A. He can appear in 7 different positions:

      $\underline{A}$ _ _ _ _ _ (this means A joins the coalition first)
     _ $\underline{A}$ _ _ _ _ (this means A joins the coalition second)
    _ _ $\underline{A}$ _ _ _ _
    _ _ _ $\underline{A}$ _ _ _
    _ _ _ _ $\underline{A}$ _ _
    _ _ _ _ _ $\underline{A}$ _
    _ _ _ _ _ _ $\underline{A}$

Keeping in mind that the blanks represent junior partners who have one vote each, in which of the 7 positions is A the pivotal voter?

b) Write A's Shapley-Shubik power index as a fraction.

c) Keeping in mind that all 7 fractions have to add up to 1, what do the Shapley-Shubik power indices of the six junior partners have to add up to?

d) Since the junior partners all have the same number of votes, they must have the same power. Find the Shapley-Shubik power index of one junior partner.

e) Write the entire Shapley-Shubik power index of this weighted voting system.

Since your computation in part (d) was dependent on the fact that each voter other than A had equal power, this shortcut will only work for weighted voting systems in which all but one voter has the same weight.