ODE

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4.4 The method of Variation of parameters

1. Second order differential equations (Normalized, standard form!).

$$y'' + P(x)y' + Q(x)y = f(x)$$

Suppose y_1 and y_2 form a fundamental set of solutions on an interval I for

$$y'' + P(x)y' + Q(x)y = 0$$

We seek functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p = u_1 y_1 + u_2 y_2$$

is a particular solution of the nonhomogeneous differential equation.

In that case,

$$y_p' = u_1 y_1' + y_1 u_1' + u_2 y_2' + y_2 u_2'$$

We can impose the additional condition on u_1 and u_2:

$$y_1 u_1' + y_2 u_2' = 0$$

That is equivalent to

$$y_p' = u_1 y_1' + u_2 y_2'$$

From there,

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Now, substitute for y_p, y_p' and y_p'' into the nonhomogeneous differential equation"

$$y_p'' + P y_p' + Q y_p = f(x)$$

which becomes:

$$u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + P u_1 y_1' + P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2 = f(x)$$

Reorganizing leads to

$$u_1'y_1' + u_1(y_1'' + Py_1' + Qy_1) + u_2'y_2' + u_2(y_1'' + Py_2' + Qy_2) = f(x).$$

and finally, one obtains a second condition on u_1 and u_2

$$y_1'u_1' + y_2'u_2' = f(x)$$

What we have now is a system of two equations involving (the derivatives of) u_1 and u_2

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= f(x) \end{aligned}$$

Observe that the determinant of the linear system in no other than the Wronskian $W(y_1, y_2) \neq 0$ by assumption. Hence, the system has a unique solution (u_1', u_2').

$$u_1' = \frac{det \begin{pmatrix} 0 & y_2 \\ f(x) & y_2' \end{pmatrix}}{W(y_1, y_2)} = -\frac{y_2 f(x)}{y_1 y_2' - y_1' y_2}$$

$$u_2' = \frac{f(x) y_1}{y_1 y_2' - y_1' y_2}$$

From u_1' and u_2', we obtain u_1 and u_2 by integration.

Example

$$y'' - 3y' + 2y = \frac{e^{3x}}{1 + e^x}$$

Notice that undetermined coefficient methods does not work in this example! Using the characteristic equation technique for instance, we find that the general solution to the associated homogeneous equation is

$$y_c = C_1 e^x + C_2 e^{2x}.$$

The method of variation of parameters tells us that we can find a particular solution

$$y_p = u_1 e^x + u_2 e^{2x}$$

by solving

$$\begin{aligned} e^x u_1' + e^{2x} u_2' &= 0 \\ e^x u_1' + 2e^{2x} u_2' &= \frac{e^{3x}}{1+e^x} \end{aligned}$$

$$u_1' = -\frac{e^{2x}}{1+e^x}$$
$$u_2' = \frac{e^x}{1+e^x}$$

$$u_1 = -(e^x+1) + \ln(e^x+1)$$
$$u_2 = \ln(e^x+1)$$

A particular solution is

$$y_p = (-(e^x+1) + \ln(e^x+1))e^x + \ln(e^x+1)e^{2x}$$

and, finally, the general solution is

$$y = C_1 e^x + C_2 e^{2x} + e^{2x}(\ln(e^x+1)) + e^x(\ln(e^x+1) - (e^x+1)).$$

Variation of parameters for higher order equations

$$y^{(n)} + P_1 y^{(n-1)} + ... + P_n y = f(x).$$

Let $y_1, ..., y_n$ be n linearly independent solutions of

$$y^{(n)} + P_1 y^{(n-1)} + ... + P_n y = 0$$

Look for $u_1, ..., u_n$ such that $u_1 y_1 + ... + u_n y_n$ is a particular solution of the nonhomogeneous equation.

For that, one solves the following linear system:

$$\begin{aligned} y_1 u_1' + ... + y_n u_n' &= 0 \\ y_1' u_1' + ... + y_n' u_n' &= 0 \\ \vdots &= \vdots \\ y_1^{(n-1)} u_1' + ... + y^{(n-1)} u_n' &= f(x) \end{aligned}$$

$$u_i' = \frac{W_i(y_1, ..., y_n)}{W(y_1, ..., y_n)}$$

where $W_i(y_1, ..., y_n)$ is the Wronskian in which the column i has been replaced by the column $(0, ...0, f(x))$.

Example

$$y''' + y' = \tan x$$

$$y''' + y' = \sec x$$

4.5 The Cauchy-Euler equation

Standard form, n^{th} order:

$$a_n x^n \frac{d^n y}{dx^n} + ... + a_1 x \frac{dy}{dx} + a_0 y = g(x).$$

Second order example:

$$x^2 y'' - 2xy' + 2y = x^3 \ln x$$

Observation: The substitution $t = \ln x$ reduces Cauchy-Euler equation to an equation with constant coefficients!

From

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}(-\frac{dy}{dt} + \frac{d^2y}{dt^2})$$

Substitution leads to

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = te^{3t}.$$

This can be solved using the method of undetermined coefficients or variation of parameters!

$$y(t) = C_1 e^{2t} + C_2 e^t + (\tfrac{1}{2}t - \tfrac{3}{4})e^{3t}$$

$$y(x) = C_1 x^2 + C_2 x + (\tfrac{1}{2}\ln x - \tfrac{3}{4})x^3$$

5.1-5.4: Applications: Spring vibrations, electric circuits problems

Second order differential equations with constant coefficients

Consider a spring with natural length L.

Suspend a mass with weight $F_g = mg$. The spring is stretched by l. Choose an orientation vector $\vec{i}$ downward.

Hooke's Law

$$mg = Kl$$

where K is the constant of the spring.

Disturb the mass m with initial position x_0 and velocity v_0.

If x denotes the displacement from equilibrium position, then

$$m\ddot{x}\vec{i} = -Kx\vec{i}$$

$$m\ddot{x} = -Kx$$

$$\ddot{x} + \lambda^2 x = 0$$

where

$$\lambda^2 = \frac{K}{m}.$$

The mass executes a free, undamped motion.

$$x(t) = C_1 \cos \lambda t + C_2 \sin \lambda t.$$

$$x(t) = A\cos(\lambda t + \Phi)$$

The simple, harmonic motion.

$\lambda = \sqrt{\frac{K}{m}}$ is the angular velocity,

$T = \frac{2\pi}{\lambda}$ is the natural period of the motion

$f = \frac{\lambda}{2\pi} = \frac{1}{T}$ is the natural frequency.

$A = \sqrt{C_1^2 + C_2^2}$ is the amplitude and Φ from $\tan\Phi = -\frac{C_2}{C_1}$ is the phase angle.

Solving

$$\lambda t + \Phi = \frac{\pi}{2}$$

$t_0 = \frac{\frac{\pi}{2} - \Phi}{\lambda}$ is the phase shift.

The motion can be graphed now!

Example #1, page 197.

Example: A 16 lb weight is placed upon the lower end of a coil spring suspended vertically from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. Determine the resulting displacement as a function of time in each of the following cases:

a) If the weight is then pulled down 4 in. below its equilibrium position and released at t=0 with initial velocity of 2 ft/sec directed downward.
b) If the weight is then pulled down 4 in. below its equilibrium position and released at t=0 with an initial velocity of 2 ft/sec directed upward.
c) If the weight is then pushed up 4 in. above its equilibrium position and released at t=0 with an initial velocity of 2 ft/sec. directed downward.

$16 = K\frac{6}{12} = \frac{K}{2}$, so $K = 32$.

$$\ddot{x} + \frac{K}{M}x = 0, \quad M = \frac{16}{32} = \frac{1}{2} \; slugs$$

$$\ddot{x} + 64x = 0$$

Finish the example!

Damped motion

$$M\ddot{x} = -Kx - \beta\dot{x}; \quad F_R = -\beta\dot{x}$$

$$\ddot{x} + \lambda^2 x + 2b\dot{x} = 0$$

where

$$2b = \frac{\beta}{M}.$$

$$\ddot{x} + 2b\dot{x} + \lambda^2 x = 0$$

Free, damped motion The motion is not necessarily periodic anymore.

$$m^2 + 2bm + \lambda^2 = 0$$

$$m = -b \pm \sqrt{b^2 - \lambda^2}.$$

$\underline{b^2 - \lambda^2 > 0}$: Over-damped motion.

$$x(t) = e^{-bt}[C_1 e^{\sqrt{b^2-\lambda^2}t} + C_2 e^{-\sqrt{b^2-\lambda^2}t}].$$

Observe the damping factor e^{-bt}!

$\underline{b^2 - \lambda^2 = 0}$: Critically damped motion.

$$x(t) = e^{-bt}(C_1 + C_2 t).$$

$\underline{b^2 - \lambda^2 = 0}$: Critically damped motion.

$$x(t) = e^{-bt}(C_1 + C_2 t).$$

$\underline{b^2 - \lambda^2 < 0}$: Under-damped motion.

$$x(t) = e^{-bt}[C_1 \cos \sqrt{\lambda^2 - b^2}\, t + C_2 \sin \sqrt{\lambda^2 - b^2}\, t].$$

Observe a periodic factor and a damping factor!

Example #2, page 208.

Forced motion

$$M\ddot{x} + \beta\dot{x} + Kx = F\cos\omega t$$

$$\ddot{x} + 2b\dot{x} + \lambda^2 x = f\cos\omega t$$

where

$$f = \frac{F}{M},\ 2b = \frac{\beta}{M},\ \lambda^2 = \frac{K}{M}.$$

Assume we are in the under-damped situation, so that

$$x_c(t) = Ae^{-bt}\cos(\sqrt{\lambda^2 - b^2}\,t + \phi).$$

We look for a particular solution using the undetermined coefficients method:

$$x_p(t) = C\cos\omega t + D\sin\omega t$$

We find that:

$$
\begin{aligned}
-\omega^2 C + 2b\omega D + \lambda^2 C &= f \\
-\omega^2 D - 2b\omega C + \lambda^2 D &= 0
\end{aligned}
$$

We find that:

$$\begin{aligned}
-\omega^2 C + 2b\omega D + \lambda^2 C &= f \\
-\omega^2 D - 2b\omega C + \lambda^2 D &= 0
\end{aligned}$$

or equivalently

$$\begin{aligned}
(\lambda^2 - \omega^2)C + 2\lambda\omega D &= f \\
-2b\omega C + (\lambda^2 - \omega)D &= 0
\end{aligned}$$

By Cramer's method for instance,

$$C = \frac{f(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}$$
$$D = \frac{2b\omega f}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}$$

$$x_p = \frac{f(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}\cos\omega t + \frac{2b\omega f}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}\sin\omega t$$

We will express x_p as

$$x_p = A\cos(\omega t + \phi)$$

$$
\begin{aligned}
A\cos\phi &= \frac{f(\lambda^2-\omega^2)}{(\lambda^2-\omega^2)^2+4b^2\omega^2} \\
-A\sin\phi &= \frac{2b\omega f}{(\lambda^2-\omega^2)^2+4b^2\omega^2}
\end{aligned}
$$

So

$$
A = \frac{f}{[(\lambda^2-\omega^2)^2+4b^2\omega^2]^{1/2}}
$$

and finally

$$
x_p(t) = \frac{f}{[(\lambda^2-\omega^2)^2+4b^2\omega^2]^{1/2}}\cos(\omega t+\phi)
$$

$$\frac{dA}{d\omega} = \frac{2(\lambda^2 - \omega^2)\omega - 4b^2\omega}{[(\lambda^2 - \omega^2)^2 + 4b^2\omega^2]^{3/2}} f$$

The critical ω's are $\omega = 0$ and $\omega^2 = \lambda^2 - 2b^2$.

$x_p(t)$ achieves the maximum amplitude when

$$\omega = \omega_R = \sqrt{\lambda^2 - 2b^2} = \sqrt{\frac{K}{M} - \frac{\beta^2}{2M^2}}$$

ω_R is called the resonance frequency.

Observe that the resonance frequency $\omega_R \leq \sqrt{\lambda^2 - b^2} = \sqrt{\frac{K}{M} - \frac{\beta^2}{4M^2}}$ is smaller than the frequency of the free motion!

The graph of $y = A(\omega)$ is called the resonance curve!

$$x(t) = x_c + x_p$$

The function $x_c(t)$ eventually becomes negligible as time goes on, this is the transient state.

$x_p(t)$ which remains forever, is called the steady state solution.

In the undamped situation, $b = 0$,

$$x_p(t) = \frac{f}{\lambda^2 - \omega^2} \cos(\omega t + \phi)$$

We can see that as ω approaches the natural frequency λ, the amplitude of the steady state blows up! This phenomenon is known as Pure resonance. It always has destructive effects on the systems.

Assignments: page 189, #10-12, Page 217 (Ross), # 4-7, Page 224, #1-3

Example: Forced motion

A mass weighting 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction and released with zero initial velocity. assuming that there is no damping, and that the mass is acted upon by an external force of $2\cos 3t$ lb, formulate the initial value problem describing the motion of the mass and solve it.

$4 = \frac{1}{8}K$ so $K = 32$

$x(0) = \frac{1}{6}$

$\dot{x}(0) = 0$

$4 = m.32$ so, $m = \frac{1}{8}$

$$\frac{1}{8}\ddot{x} + 32x = 2\cos 3t$$

$$\begin{aligned} \ddot{x} + 256x &= 16\cos 3t \\ x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned}$$

Solving the above initial value problem leads to

$$x(t) = \frac{151}{1482}\cos 16t + \frac{16}{247}\cos 3t.$$

Analogous systems

Mass on a spring:

$$m\ddot{x} + \beta\dot{x} + Kx = f(t)$$

Inductor-Resistor-Capacitor (L-R-C) series circuit:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t)$$

($E(t)$ is the electric potential)

Put $E(t) = E_0 \sin \omega t$

Write the circuit equation as

$$LI'' + RI' + \frac{1}{C} I = \omega E_0 \cos \omega t$$

Solving:

$$I(t) = I_{tr} + I_{sp}$$

where

$$\lim_{t \to +\infty} I_{tr} = 0$$

$$I_{sp} = \textit{Steady periodic solution}$$

$$I_{sp} = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

$$\alpha = \tan^{-1} \frac{\omega RC}{1 - LC\omega^2}, \ 0 \le \alpha \le \pi$$

$$Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

(in Ohms) is called the impedance of the circuit. The amplitude of the signal is

$$I_0 = \frac{E_0}{Z}$$

Electrical resonance

$I_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}}$ attains its maximum when

$$\omega = \frac{1}{\sqrt{LC}}$$

This is the resonance frequency!. Tuning a radio receiver consists essentially in modifying the value of C so as to match the frequency of the incoming radio signal!

Examples

Solve

$$\frac{dy}{dx} - 2xy = 0$$

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n c_n x^{n-1}$$

and

$$\frac{dy}{dx} - 2xy = \sum_{n=0}^{\infty} n c_n x^{n-1} - 2\sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$\sum_{m=0}^{\infty}(m+1)c_{m+1}x^m - 2\sum_{m=1}^{\infty} c_{m-1}x^m = 0$$

$$c_1 + \sum_{m=1}^{\infty}[(m+1)c_{m+1} - 2c_{m-1}]x^m = 0$$

$c_1 = 0$ and $(m+1)c_{m+1} - 2c_{m-1} = 0$ which implies that

$$c_1 = 0 \textit{ and } c_{m+1} = 2\frac{c_{m-1}}{m+1}$$

for $m \geq 2$.

We will use the recurrence formula for c_m to generate all coefficients in the power series.

$$
\begin{aligned}
c_0 & \quad \textit{arbitrary} \\
c_1 &= 0 \\
c_2 &= 2\frac{c_0}{2} = c_o \\
c_3 &= 2\frac{c_1}{3} = 0 \\
c_4 &= 2\frac{c_2}{4} = \frac{c_0}{2} \\
c_5 &= 0
\end{aligned}
$$

In general,

$$c_{2n+1} = 0$$

for $n = 0, 1, 2...$ and

$$\begin{aligned} c_{2n} &= 2\frac{c_{2(n-1)}}{2n} \\ &= 2\frac{c_{2(n-2)}}{(n-2)(n-1)} = \frac{c_{2(n-2)}}{n(n-1)} \\ \vdots &= \vdots \\ &= \frac{c_0}{n!} \end{aligned}$$

$$y = c_0\left(1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{m!}\right) = c_0\left(\sum_{m=0}^{\infty} \frac{\left(x^2\right)^m}{m!}\right)$$

$$y = c_0 e^{x^2}.$$

$$y = c_0 \left(1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{m!} \right) = c_0 \left(\sum_{m=0}^{\infty} \frac{\left(x^2\right)^m}{m!} \right)$$

$$y = c_0 e^{x^2}.$$

Check this solution using the separable equation technique!

Example 2

$$(x-1)y'' - xy' + y = 0, \;\; y(0) = -2, \; y'(0) = 6$$

$$y = \sum_{n\geq 0} c_n x^n$$

$$y' = \sum_{n\geq 0} nc_n x^{n-1}$$

$$y'' = \sum_{n\geq 0} n(n-1)c_n x^{n-2}$$

$$\begin{aligned} 0 &= (x-1)y'' - xy' + y = xy'' - y'' - xy' + y \\ &= \sum_{n\geq 0} [(n+1)nc_{n+1} - (n+2)(n+1)c_{n+2} + (1-n)c_n]x^n \end{aligned}$$

So

$$c_{n+2} = \frac{(1-n)c_n + (n+1)nc_{n+1}}{(n+2)(n+1)}$$

$$c_0 = y(0) = -2$$

$$c_1 = y'(0) = 6$$

Using the above recurrence formula, we see that

$c_2 = -1,\ c_3 = -\frac{1}{3},\ c_4 = -\frac{1}{4.3},\ c_5 = -\frac{1}{5.4.3}$

Claim: for $n \geq 2$, $c_n = -\frac{2}{n!}$.

proof

True for c_2 and c_3. Assume it is true for $n-1$ and $n-2$, then

$$\begin{aligned} c_n &= \frac{(1-n)c_{n-2} + (n-1)(n-2)c_{n-1}}{n(n-1)} \\ &= \frac{(1-n)\frac{-2}{(n-2)!} + (n-1)(n-2)\frac{-2}{(n-1)!}}{n(n-1)} \\ &= -\frac{2}{n!} \end{aligned}$$

$$\begin{aligned} y &= -2 + 6x - 2\sum_{n\geq 2} \frac{x^n}{n!} \\ &= 8x - 2 - 2x - 2\sum_{n\geq 2} \frac{x^n}{n!} \\ &= 8x - 2(\sum_{n\geq 0} \frac{x^n}{n!}) \\ y &= 8x - 2e^x \end{aligned}$$

Check by substitution that this is indeed the solution to the initial value problem.

Example 2

$$(x-1)y'' - (2-x)y' + y = 0, \;\; y(0) = 2, \; y'(0) = -1$$

$$c_0 = y(0) = 2, \; c_1 = y'(0) = -1$$

The recurrence relation is:

$$c_{n+2} = \frac{c_n + (n-2)c_{n+1}}{n+2}$$

$$y = 2 - x + 2x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{10} + \frac{x^6}{20} + \frac{x^7}{140} + \ldots$$

Series solutions around ordinary points

$$y'' + P(x)y' + Q(x)y = 0$$

The standard form for a second order linear differential equation.

Definition

A point x_0 is said to be an ordinary point for the above differential equation if $P(x)$ and $Q(x)$ are analytic at x_0; that is, both have power series in $(x - x_0)$ with positive radius of convergence.

A point that is not an ordinary point is said to be a singular point.

Note: For power series solutions at an ordinary point, the radius of convergence is at least equal to the distance to the nearest singular point.

Example

$$(x^2 - 1)y'' + 4xy' + \frac{2}{x^2 - 1}y = 0$$

takes the standard form

$$y'' + \frac{4x}{x^2 - 1}y' + \frac{2}{(x^2 - 1)^2}y = 0$$

1 and -1 are singular points for this equation. Any other point is a regular point, according to the above definition.

Theorem

If $x = x_0$ is an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$, we can always find two linearly independent power series solutions of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Example

Find two linearly independent solutions for

$$y'' - xy' + 2y = 0$$

Example

Find two linearly independent solutions for

$$y'' - xy' + 2y = 0$$

Look for

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Then:

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Therefore, $y'' - xy' + 2y = 0$ implies that

$$\begin{aligned}
0 &= \textstyle\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n \\
&= \textstyle\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} 2c_n x^n - \sum_{n=1}^{\infty} nc_n x^n \\
&= 2c_2 + 2c_0 + \textstyle\sum_{n=1}^{\infty}[(n+2)(n+1)c_{n+2} + (2-n)c_n]x^n
\end{aligned}$$

We deduce that

$$c_2 = -c_0$$

$$c_{n+2} = \frac{(n-2)c_n}{(n+2)(n+1)}$$

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$$c_2 = -c_0$$

$$c_{n+2} = \frac{(n-2)c_n}{(n+2)(n+1)}$$

c_0: Arbitrary

$c_2 = -c_0$

$c_4 = 0$

$$c_{2n} = 0,\ n > 2$$

c_1 is also arbitrary.

$$c_{2n+1} = \frac{(2n-3)c_{2n-1}}{(2n+1)(2n)},\ 2n+1 > 3$$

$c_3 = -\frac{c_1}{3.2}$

$$y_1 = c_0 - c_0 x^2 = c_0(1 - x^2)$$

$$y_2 = c_1 x - \frac{c_1}{6} x^3 + .. = c_1(x - \frac{1}{6} x^3 + ...)$$

Another example

$$y'' - xy' - y = 0, \ x_0 = 1$$

Find two linearly independent power series solutions.

Look for

$$y = \sum_{n=0}^{\infty} c_n (x-1)^n$$

for

$$y'' - (x-1)y' - y' - y = 0$$

6.2. Solutions about singular points: Frobenious Method

Definition

For a differential equation

$$Py'' + Qy' + Ry = 0$$

A singular point x_0 is said to be a regular singular point if $(x - x_0)\frac{Q}{P}$ and $(x - x_0)^2\frac{R}{P}$ are analytic at x_0. Otherwise, the regular point x_0 is said to be irregular singular.

Example: The Euler Equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

$x = 0$ is a regular singular point. As an alternate method of solution, we look for

$$y = x^r$$

Then

$$x^r(x^r)''_\alpha x(x^r)' + \beta x^r = 0 = x^r(r(r-1) + \alpha r + \beta)$$

Any solution of

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

leads to a solution $y = x^r$ of the Euler equation.

Real, Distinct Roots

If r_1 and r_1 are the 2, real distinct roots, then

$$y = Ax^{r_1} + Bx^{r_2}$$

Example: $2x^2y'' + 3xy' - y = 0, \; x > 0$

Double Roots

$$F(r) = (r - r_1)^2$$

In this case $y_1 = x^{r_1}$ is a solution and the method of reduction of order shows that

$$y_2 = x^{r_1} \ln x$$

is also a solution. (Check this directly!)

Example:

$$x^2 y'' + 5xy' + 4y = 0, \ x > 0$$

Complex-Conjugate Roots

$$r_1 = \lambda + i\mu, \;\; r_2 = \lambda - i\mu$$

$$z_1 = x^{\lambda + i\mu} = e^{(\lambda + i\mu)\ln x} = x^{\lambda}(\cos(\mu \ln x) + i \sin(\mu \ln x))$$

$$z_2 = x^{\lambda}(\cos(\mu \ln x) - i \sin(\mu \ln x))$$

$$y_1 = \frac{1}{2}(z_1 + z_2) = x^\lambda \cos(\mu \ln x)$$

and

$$y_2 = -\frac{1}{2i}(z_1 - z_2) = x^\lambda \sin(\mu \ln x)$$

are two linearly independent real solutions.

Example:

$$x^2 y'' + xy' + y = 0$$

More Generally

Example:

$$2x^2y'' - xy' + (1+x)y = 0$$

0 is a regular singular point. We look for

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Substituting into the differential equation and grouping like terms from lowest power of x to higher powers:

$$F(r)x^{r+k} + \sum_{n\geq 1} [G(r,n)] x^{n+r+k} = 0$$

Solve the indicial equation

$$F(r) = 0$$

and obtain recurrence relations from

$$G(r, n) = 0$$

If $r_1 - r_2$ is not an integer, we always obtain two linearly independent solutions.

Example 2

$$x^2y'' + (x^2 + 4x)y' + (2x + 2)y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum (n + r)c_n x^{n+r-1}, \quad y'' = \sum (n + r)(n + r - 1)c_n x^{n+r-2}$$

$$x^2 y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r+1} = \sum_{m=1}^{\infty} (m-1+r) c_{m-1} x^{m+r}$$

$$2xy = \sum_{n=0}^{\infty} 2 c_n x^{n+r+1} = \sum_{m=1}^{\infty} 2 c_{m-1} x^{m+r}$$

$$\sum(n+r)(n+r-1)c_n x^{n+r} + \sum(n+r)c_n x^{n+r+1} + \sum 4(n+r)c_n x^{n+r} + \sum 2c_n x^{n+r+1} + \sum 2c_n x^{n+r} = 0$$

$$c_0(r(r-1)+4r+2)x^r + \sum_{m=1}^{\infty}\left[[(m+r)(m+r+3)+2]c_m + (m+r+1)c_{m-1}\right]x^{m+r} = 0$$

$$F(r) = r(r+3)+2 = 0 = r^2+3r+2$$

$$c_m = -\frac{(m+r+1)c_{m-1}}{(m+r)(m+r+3)+2}$$

$$r_1 = -2, \; r_2 = -1$$

Work with the smaller root -2 first:

$$c_n = -\frac{c_{n-1}}{n}$$

Inductively, we see that

$$c_n = \frac{(-1)^n c_0}{n!}$$

This leads to

$$y_1 = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n-2} = c_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = c_0 x^{-2} e^{-x}$$

Form $r = -1$, we obtain

$$c_n = -\frac{c_{n-1}}{n+1}$$

Inductively:

$$c_n = (-1)^n \frac{c_0}{(n+1)!}$$

$$Y_2 = c_0 \sum \frac{(-1)^n}{(n+1)!} x^{n-1} = -c_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} = -c_0 x^{-2} (e^{-x} - 1)$$

Check directly that x^{-2} and $x^{-2} e^{-x}$ are solutions!

The general solution is

$$y = Ax^{-2} e^{-x} + Bx^{-2}$$

6.3 Bessel's Equation and Bessel's functions

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

Bessel's Equation of order p.

Any solution is called a Bessel function of order p. Theses functions occur in connection with problems of Physics and Engineering.

The case $p = 0$

$$xy'' + y' + xy = 0$$

0 is a regular, singular point.

Look for

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$r^2 c_0 x^{r-1} + (1+r) c_1 x^r + \sum_{n=2}^{\infty} [(n+r)^2 c_n + c_{n-2}] x^{n+r-1} = 0$$

The indicial equation is

$$r^2 = 0$$

with double root $r_1 = 0 = r_2$.

Then

$$(1+r)^2 c_1 = 0$$

and

$$(n+r)^2 c_n + c_{n-2} = 0, \;\; n \geq 2$$

$$r = 0 \Rightarrow c_1 = 0$$

$$r = 0 \Rightarrow n^2 c_n + c_{n-2} = 0 \Rightarrow \;\; c_n = -\frac{c_{n-2}}{n^2}$$

$$c_{2n+1} = 0$$
$$c_{2n} = \frac{(-1)^n c_0}{(n!)^2 2^{2n}}, \;\; n \geq 1$$

$$y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (\frac{x}{2})^{2n} = y_1(x).$$
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (\frac{x}{2})^{2n}$$

Bessel Function of the first kind of order zero.

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \ldots$$

A second solution must be of the form (See Theorem 6.3)

$$y = x \sum^{\infty} c_n^* x^n + J_0(x) \ln x$$

8.3. Successive approximations

$$y' = f(x, y),\ y(x_0) = y_0.$$

Picard's Method

$y = \phi(x)$, the d.e. is just specifying the slope of the tangent line to the graph of the solution!

Zeroth approximation: $\phi_0 = y_0$.

First approximation: ϕ_1 (satisfying a different d.e.)

$$\phi_1'(x) = f(x, \phi_0(x)),\ \phi_1(x_0) = y_0.$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0)dt$$

Second approximation: ϕ_2. (Satisfying a different d.e.)

$$\phi_2'(x) = f(x, \phi_1), \;\; \phi_2(x_0) = y_0.$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))dt$$

$\vdots$

n^{th} approximation:

$$\phi_n(x) = \int_{x_0}^{x} f(t, \phi_{n-1}(t))dt$$

One has a sequence of functions $\phi_0, \phi_1, ..., \phi_n, ...$ The exact solution is given by:

$$\phi = \lim_{n\to\infty} \phi_n$$

Picard used this method to prove existence of solutions!

Observe that

$$\phi'(x) = \lim_{n\to\infty} \phi_n'(x) = \lim_{n\to\infty} f(x, \phi_{n-1}(x)) = f(x, \phi).$$

Example

$$y' = xy, \quad y(0) = 1$$

Let

$$y = \phi(x)$$

$$\phi_0 = 1$$

$$\phi_1 = 1 + \int_0^x t dt = 1 + \frac{x^2}{2}.$$

$$\phi_2 = 1 + \int_0^x t(1 + \frac{t^2}{2}) dt = 1 + \frac{x^2}{2} + \frac{(\frac{x^2}{2})^2}{2}$$

$$\phi_3 = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!}$$

where $u = \frac{x^2}{2}$.

$\vdots$

$$\phi_n(x) = \sum_{i=1}^{n} \frac{u^i}{i!}$$

Where $u = \frac{x^2}{2}$.

$$\phi(x) = \lim_{n \to \infty} \phi_n(x) = e^{\frac{x^2}{2}}.$$

Approximating the solution of

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Let $x_1 = x_0, x_2 = x_0 + h, ...$

$$x_N = x_{N-1} + h$$

If $\phi(x)$ is the exact solution, let $\phi(x_1), ..., \phi(x_N)$ be the evaluation of ϕ at points x_k.

A numerical method will use the IVP to estimate $\phi(x_k),\ k = 1, 2, ..., N$.

Let $y_1, y_2, ..., y_N$ be approximations to $\phi(x_1), , \phi(x_2), ..., \phi(x_N)$. A one-step method uses y_{k-1} to find y_k using the differential equation. The method has also an alternate name of “starting method".

The multi-step methods (using several previous approximations to find y_k) are also known as "continuing methods".

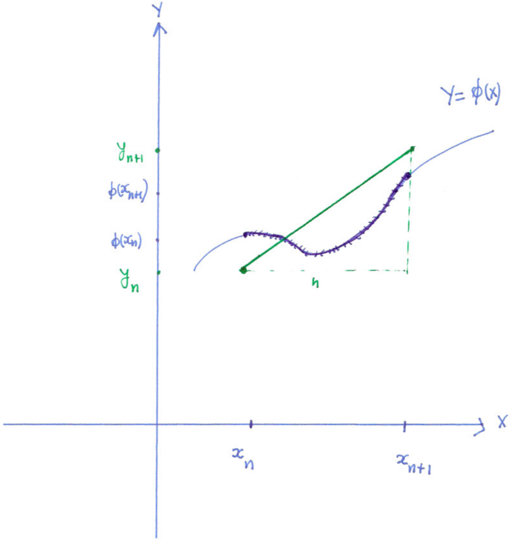
The Euler Method

ϕ the exact solution of $y' = f(x, y),\ y(x_0) = y_0$.

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Geometrically, the segment of then graph of $y = \phi(x)$ between $(x_n, \phi(x_n))$ and $(x_{n+1}, \phi(x_{n+1}))$ is replaced by the line segment joining (x_n, y_n) and $(x_{n+1}, y_n + hf(x_n, y_n))$.

$y_0 = \phi(x_0)$



Y
$Y = \phi(x)$
y_{n+1}
$\phi(x_{n+1})$
$\phi(x_n)$
y_n
h
X
x_n
x_{n+1}

Example

$$y' = 2x + y; \quad y(0) = 1$$

Use $h = 0.2$.

$x_0 = 0$	$x_1 = 0.2$	$x_3 = 0.4$	$x_3 = 0.6$	$x_4 = 0.8$
$y_0 = 1$	$y_1 = \frac{12}{10}$	$y_2 = \frac{152}{100}$	$y_3 = \frac{1984}{1000}$	$y_4 = \frac{26168}{10000}$