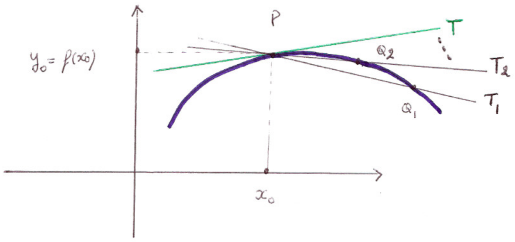
2.1 Tangent lines and rates of change



The point $Q(x, f(x))$ moves toward $P(x_0, f(x_0))$ along the graph as $x \to x_0$. The slope m_x of T_x is given by:

$$m_x = \frac{f(x) - f(x_0)}{x - x_0}$$

The limiting case, the tangent line T has slope given by:

$$m_{x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Example: Find the equation of the tangent line to the curve $y = x^3 + 2$ at $x_0 = 1$.

Solution:

The tangent line has slope

$$\begin{aligned} m &= \lim_{x \to 1} \frac{x^3 + 2 - 3}{x - 1} \\ &= \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3 \end{aligned}$$

The point where the tangent line intersect the curve is $(1, 3)$. So the equation for the tangent line at $x = 1$ is

$$y - 3 = 3(x - 1)$$

or after simplifications,

$$y = 3x$$

We shall use the notation $f'(x_0)$ for the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$f'(x_0)$ is called the derivative of $f(x)$ at x_0.

Remark: If two quantities x and y are related functionally by $y = f(x)$, then the average rate of change of y with respect to x over an interval$[x_0, x_1]$ is is given by the slope of the secant line joining $(x_0, f(x_0))$ and $(x_1, f(x_1)$. That is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If defined, the derivative $f'(x_0)$ gives the instantaneous rate of change of y with respect to x at x_0.

For example, the instantaneous rate of change of distance covered with respect to time is known as the speed for a vehicle in motion.

2.2 The derivative function

The function f' defined for each x by the derivative at x is called the derivative of f with respect to x

It is given by the formula:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The notation

$$\frac{d}{dx}[f(x)]$$

will also be used for $f'(x)$

Example Find $f'(x)$ for $f(x) = 2x^2 + 1$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2(x+h)^2 + 1 - (2x^2 + 1)}{h} \\ &= \lim_{h \to 0} 4x + 2h = 4x \end{aligned}$$

So $f'(x) = 4x$.

Theorem

If $f(x)$ is differentiable at x_0, that is, $f'(x_0)$ is defined, then f is continuous at x_0.

Proof: Need to prove $\lim_{h\to 0}(f(x_0+h)-f(x_0))=0$, from which will follow $\lim_{h\to 0} f(x_0+h)=f(x_0)$, the required condition for continuity at x_0.

$$\begin{aligned}
\lim_{h\to 0}(f(x_0+h)-f(x_0)) &= \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}h\right) \\
&= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\lim_{h\to 0} h \\
&= f'(x_0).0 = 0
\end{aligned}$$

However, there are functions which are continuous at some points without being differentiable there!.

example

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to 0^-} f(x)$$

so $f(x) = |x|$ is continuous at $x = 0$.

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

hence,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

doesn't exist and therefore, $f'(0)$ is not defined.