Calculus II

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5.4. Sigma notation; The definition of area as limit

Assignment: page 350, #11-15, 27, 29, 37, 38, 48.

$$1 + 2 + 3 = 6$$

How about

$$1 + 2 + ... + 100 = ?$$

In Sigma notation

$$\sum_{i=1}^{100} i$$

Notice

$$\sum_{i-1}^{100} = 100 + 99 + ... + 1$$

So

$$2\sum_{i=1}^{100} = (100 + 1) + (99 + 2) + ... + (1 + 100) = 101 \times 100$$

Therefore

$$\sum_{i=1}^{100} = \frac{101 \times 100}{2}$$

In general, we have the following formulas:

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$$

Also:

$$\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{(n+1)n}{2}\right)^2$$

The area Problem:

Given the graph of a positive function

$$y = f(x)$$

compute the area under the graph, above the x-interval [a,b].

Solution: Subdivide the interval [a,b] by markings

$$x_0 = a, x_1 = x_0 + \Delta x_1, ..., x_i = x_{i-1} + \Delta x_i, ..., x_n = b$$

The area of the region above $[x_{i-1}, x_i]$ is approximately the area of a rectangle with base $[x_{i-1}, x_i]$ and height $f(c_i)$ where c_i is some number satisfying

$$x_{i-1} \leq c_i \leq x_i.$$

Ultimately, the area A of the region mentioned at the beginning is approximately

$$A_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

In the end, the actual area is obtained by taking the limit of A_n when $n \to \infty$.

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Assignment: page 360, #14-17.

For an arbitrary function $f(x)$, not necessarily positive, one still has the summations

$$A_n = \sum_{1}^{n} f(c_i)\Delta x_i$$

but they are not approximate areas anymore, we call them just Riemann sums of $f(x)$, the limit, if it exists, is called the Riemann integral of f over the interval $[a, b]$. The standard notation is

$$\int_a^b f(x)dx.$$

The above is also known as the Definite integral of f over the interval $[a, b]$.

Theorem

If a function f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$, that is, the definite integral $\int_a^b f(x)dx$ exists.

Basic Properties of the integral

- $\int_a^b C\,dx = C(b-a)$ for any constant C
- $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

- For $a < c < b$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

- If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

The Mean Value Theorem for integrals

Suppose $f(x)$ is continuous on the interval $[a, b]$

If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

and

$$m \leq \frac{1}{(b-a)} \int_a^b f(x)dx \leq M$$

so, by the intermediate value Theorem (Calculus I), there exists a $\bar{x}$, $a \leq \bar{x} \leq b$ such that

$$f(\bar{x}) = \frac{1}{(b-a)} \int_a^b f(x)dx.$$

Assignment: page 373, #20-25, 32, 34, 60-64.

Part I

Let f be continuous on an open interval I and let $a \in I$. If F is defined by

$$F(x) = \int_a^x f(t)dt$$

at all $x \in I$, then

$$F'(x) = f(x)$$

at each $x \in I$.

Proof

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}[\int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt] \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \\
&= \lim_{h\to 0} f(\bar{t}),\ x \le \bar{t} \le x+h \\
&= f(x)
\end{aligned}
$$

where $x \le \bar{t} \le x+h$ is provided by the mean value Theorem for integral.

Part II

$$\int_a^b f(x)dx = G(b) - G(a)$$

for any antiderivative $G(x)$ of $f(x)$.

Proof

From Part I, an antiderivative of $f(x)$ is given by

$$A(x) = \int_a^x f(t)dt$$

Suppose $G(x)$ is any other antiderivative of $f(x)$. Then

$$A(x) = G(x) + C$$

for some constant C. Also, notice that $A(a) = 0$, therefore, $C = -G(a)$ and $A(b) = \int_a^b f(x)dx$.

It follows that

$$\int_a^b f(x)dx = A(b) = G(b) + C = G(b) - G(a).$$

Examples: Compute $\int_1^2 x^2+1\ dx$, $\int_0^{\frac{\pi}{3}} \sin 3x\ dx$, $\int_4^8 \frac{1}{x}\ dx$, $\int_6^{11} \frac{1}{x-1}\ dx$.

Compute the average value of the cross sectional area of a disc of radius 1.

Answer: $\bar{A} = \frac{1}{1-0}\int_0^1 A(x)dx$ where $A(x)$ is the cross sectional disc at radial distance x from the center.

Assignment: page 383, #14-16, 34-36, 41, 43.

Finding position and velocity.

$D(t)$: the position; $V(t) = D'(t)$: the velocity and $A(t) = V'(t)$ represents the acceleration.

$s(t)$ will represent the distance traveled, not the same as the displacement. $s'(t) = v(t)$ where $v(t) = |V(t)|$ is the speed.

Equivalent integral formula:

$D(t) = \int V(t)dt$; $V(t) = \int A(t)dt$; $s(t) = \int v(t)dt$.

Suppose that a particle moves on a coordinate line so that its velocity at time t is $V(t) = t^2 - 2t$ m/s.

a) Find the displacement of the particle during the time interval $0 \leq t \leq 5$

b) Find the distance traveled by the particle during the time interval $0 \leq t \leq 5$.

answer:

a)

$$D(5) - D(0) = \int_0^5 V(t)dt = \frac{50}{3}$$

b)

$$s(5) - s(0) = \int_0^5 v(t)dt = \int_0^2 -t(t-2)dt + \int_2^5 t(t-2)dt = \frac{58}{3}.$$

Uniformly accelerated motion

$A(t) = A$ is constant.

Let $D_0 = D(0),\ V_0 = V(0)$. Then

$$V(t) = \int A(t)dt = At + C$$

$V_0 = C$ implies that

$$V(t) = At + V_0$$

$$D(t) = \int V(t)dt = A\frac{t^2}{2} + V_0 t + K$$

$D_0 = K$ implies that

$$D(t) = \frac{A}{2}t^2 + V_0 t + D_0.$$

Example: Falling object.

$A = -g$ where g is the gravitational constant ($9.8m/sec^2$ or $32ft/sec^2$).

A projectile is fired vertically upward from ground level with initial velocity v_0 of 16 ft/s.

a) How long will it take for the projectile to hit the ground?
b) How long will the projectile be moving upward?

Answer: a) t=1

b) $V(t) = 0$ when $t = \frac{1}{2}$.

5.9. Evaluating definite integrals by substitution

Assignment: page 394, #10-15, 35-40.

Use both ways to evaluate the following integrals:

$\int_{-3}^{0} x\sqrt{1-x}dx$, $\int_{0}^{\frac{\pi}{6}} 2\cos 3x dx$, $\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+9}}$,

$\int_{\frac{\pi}{2}}^{\pi} 6\sin x(\cos x+1)^5 dx$.

$\int_{-1}^{1} \frac{x^2 dx}{\sqrt{x^3+9}}$

$\int_{\frac{\pi}{2}}^{\pi} 6\sin x(\cos x+1)^5 dx$

5.10. Logarithmic functions from the integral point of view

Assignment: page 406, #3, 4, 7, 11, 12.

Starting point: Define $\ln x$ so that $\ln 1 = 0$ and $\frac{d}{dx} \ln x = \frac{1}{x}$.

Definition

$$\ln x = \int_1^x \frac{dt}{t}, \;\; x > 0.$$

Geometrically, $\ln x$ is the signed area to the right of $x = 1$, under the graph of $y = \frac{1}{x}$.

The graph of $y = \ln x$.

Notice $\lim_{x\to+\infty} \ln x = +\infty$ and $\lim_{x\to 0} \ln x = -\infty$.

The number e

Since $\ln' x > 0$, $\ln x = 1$ has a unique solution, this solution is denoted by e and is approximated by $e \simeq 2.71828....$, it is irrational!.

Laws of logarithms

- $\ln(xy) = \ln x + \ln y, \; \textit{for } x, y > 0$
- $\ln(\frac{1}{x}) = -\ln x$
- $\ln(\frac{x}{y}) = \ln x - \ln y$
- $\ln(x^r) = r \ln x$

Proof of laws of logarithms

$\frac{d}{dx}(\ln(xy) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}\ln x$ so $\ln(xy) - \ln x = c$ a constant independent of x.

Put $x = 1$ and observe that $\ln y - 0 = c$ so that

$$\ln(xy) - \ln x = \ln y$$

Equivalently

$$\ln(xy) = \ln x + \ln y.$$

$\frac{d}{dx}\ln(\frac{1}{x}) = x(-\frac{1}{x^2}) = -\frac{1}{x} = -\frac{d}{dx}\ln x$. So $\ln(\frac{1}{x}) + \ln x = c$ a constant independent of x.

Put $x = 1$ and observe that $0 + 0 = c$, thus

$$\ln(\frac{1}{x}) = -\ln x.$$

Next law:

$$\ln(\frac{x}{y}) = \ln(x.\frac{1}{y}) = \ln x - \ln y.$$

Next:

$\frac{d}{dx}(x^r) = rx^{r-1}$.

$\frac{d}{dx}(\ln x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d}{dx}(r \ln x)$. Hence, $\ln x^r - r \ln x = c$, a constant independent of x. Put $x = 1$ and observe that $0 - 0 = c$, so $\ln x^r - r \ln x = 0$ or equivalently:

$$\ln x^r = r \ln x.$$

The natural exponential

$\ln x$ is strictly increasing for $x > 0$, so it has an inverse function e^x defined by:

$$x = \ln y \iff y = e^x.$$

One has the identities $\ln(e^x) = x$ for all x and $e^{\ln x} = x$ for all $x > 0$. Some other facts about e^x are:

$$e^0 = 1, \quad \lim_{x \to +\infty} e^x = +\infty, \quad \lim_{x \to -\infty} e^x = 0.$$

Laws of exponents

- $e^x e^y = e^{x+y}$
- $e^{-x} = \frac{1}{e^x}$
- $(e^x)^r = e^{rx}$.

Proof of the laws of exponents

$$\ln(e^x e^y) = \ln e^x + \ln e^y = x + y = \ln(e^{(x+y)}).$$

Since $\ln x$ is one-to-one, one concludes that $e^x e^y = e^{x+y}$.

Derivatives and integrals

For any positive number $a \neq 1$, one has

$$a^u = (e^{\ln a})^u = e^{u \ln a}.$$

So,

$$\frac{d}{dx}(a^{u(x)}) = \frac{d}{dx} e^{u(x)\ln a} = e^{u \ln a}. \ln a u'(x) = (\ln a) a^u \frac{du}{dx}.$$

$$\int a^x dx = \int e^{x \ln a} dx = \frac{e^{x \ln a}}{\ln a} + C.$$

Examples

$\frac{d}{dx}(3^{x^3}) = 3^{x^3}. \ln 3.3x^2 = 3\ln 3x^2 3^{x^3}.$

$$\begin{aligned}
\int x 10^{x^2} dx &= \int x e^{\ln 10 x^2} dx \\
&= \frac{1}{2\ln 10}\int e^u du = \frac{1}{2\ln 10} e^{\ln 10 x^2} + C \\
&= \frac{10^{x^2}}{2\ln 10} + C
\end{aligned}$$

where $u = \ln 10 x^2$, $du = 2\ln 10 x dx$. (Integration by substitution).

Assignment: page 419, #3,4,6,8,10,17.

Assumptions: $f(x) \geq g(x)$ for all $x \in [a, b]$

R denotes the region between the graphs of $y = f(x)$ and $y = g(x)$.

The area A of the region R is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

1. Find the area of the region enclosed by the following curves:

$y = x^2$ and $y = 4x$.

<u>Solution</u>: The region is bounded on top by $y = 4x$, on bottom by $y = x^2$, on left by $x = 0$ and on right by $x = 4$. This is determined by finding all intersection point between $y = 4x$ and $y = x^2$, as well as sketching the curves.

$$A = \int_0^4 (4x - x^2)dx = 2x^2 - \frac{x^3}{3}\Big|_{x=0}^{4} = \frac{32}{3}.$$

2. Find the area enclosed by $y = x^3 - 4x^2 + 3x$, $y = 0$, $x = 0$, and $x = 3$

Solution: The region is made of 2 pieces, one above the interval $[0, 1]$ the other below $[1, 3]$. The area is given by

$$A = \int_0^1 (x^3 - 4x^2 + 3x)dx + \int_1^3 -(x^3 - 4x^2 + 3x)dx = 3.08$$

3. $y = x^3 - 2x^2$, $y = 2x^2 - 3x$, $x = 0$ and $x = 3$.

$(0,0)$, $(1,-1)$ and $(3,9)$ are intersection points. The area is given by

$$A = \int_0^1 (x^3 - 2x^2) - (2x^2 - 3x)dx + \int_1^3 (2x^2 - 3x) - (x^3 - 2x^2)dx = 2.2$$

4. Find the area between the curve $y = \sin x$ and the line segment that joins the points $(0,0)$ and $(\frac{5\pi}{6}, \frac{1}{2})$ on the curve.

$$A = \int_0^{\frac{5\pi}{6}} (\sin x - \frac{3}{5\pi}x)dx$$

6.2 Volume by slicing; Disc and washers

Assignment: page 429, #7,8,10,40-45.

Imagine a solid object lying along the X axis over the interval $[a, b]$. Perform a vertical uniform slicing into n slices, each of thickness Δx and sectional area $A(x_1)$, $i = 1, 2, ..., n$. As long as the slicing is fine, each slice has volume more or less given by $\Delta V_i = A(x_i)\Delta x$ and the solid itself has volume approximated by

$$V_n = \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(x_i)\Delta x$$

This is a Riemann sum. Let the number of slice n go to infinity and obtain the volume of the solid as

$$V = \lim_{n \to \infty} V_n = \int_a^b A(x)dx.$$

In many instances, an expression for the function $A(x)$ can be easily determined. For example for solid of revolution, slices turn out to be **discs** or **washers**, depending on the solid being plain or hollow.

Examples

1. Compute the volume of a solid cylinder of radius r and height h.

Solution

For $0 \leq x \leq h$, a slice has area $A(x) = \pi r^2$, so the volume of the cylinder is

$$V = \int_0^h \pi r^2 dx = \pi r^2 x \Big|_0^h = \pi r^2 h$$

as expected and known.

2. Compute the volume of a hollow cylinder with inner radius r and outer radius R and height h.

Here, each slice is a washer whose area is

$$A(x) = \pi R^2 - \pi r^2.$$

So the volume is

$$V = \int_0^h (\pi R^2 - \pi r^2) dx = \pi(R^2 - r^2)h.$$

In general, when a region bounded by the graph of $y = f(x)$ for $a \leq x \leq b$, is revolved about the X-axis, the solid thus generated has volume

$$V = \int_a^b \pi f^2(x) dx.$$

For a hollow solid generated using $f(x) \geq g(x)$, the volume formula is

$$V = \pi \int_a^b (f^2(x) - g^2(x)) dx.$$

3. Find the volume of the solid that results when the region enclosed by $y = \sec x$, $x = \frac{\pi}{4}$, $x = \frac{\pi}{3}$ and $y = 0$ is revolved about the x-axis.

Here $A(x) = \pi \sec^2 x$ and therefore,

$$V = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \pi \sec^2 x dx = \pi \tan x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \pi(\sqrt{3} - 1)$$

4. The region bounded by $y = x^2$ and $x = y^2$ is revolved about the x axis. The solid thus generated has volume

$$V = \int_0^1 \pi((\sqrt{x})^2 - (x^2)^2)dx = \pi \int_0^1 (x - x^4)dx = \frac{3\pi}{10}.$$

6.3. Volume by cylindrical shells

Assignment: page 436, #1,2,8-11, 15,16,30-32.

We perform a uniform vertical drilling into n cylindrical shells. Provided the drilling is fine enough, each cylindrical shell has volume approximated by

$$\Delta V_i = \pi r_i^2 h_i - \pi r_{i-1}^2 h_i = \pi h_i (r_i + r_{i-1})(r_i - r_{i-1}) = 2\pi h_i (\frac{r_i + r_{i-1}}{2})\Delta r$$

The volume of the solid is itself approximated by a Riemann sum

$$V \simeq \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} 2\pi h_i (\frac{r_i + r_{i-1}}{2})\Delta r$$

Actual volume is obtained by taking the limit when $n \to \infty$ and it is given by the definite integral

$$V = 2\pi \int_a^b rh(r)\, dr$$

Here, a and b are the inner and outer radius in the solid.

Examples

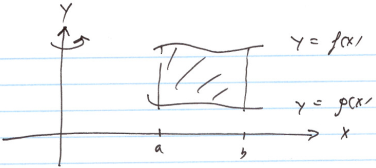
Consider the region bounded by the curves $y = \cos x^2$, $x = 0$, $x = \frac{1}{2}\sqrt{\pi}$ and $y = 0$. Revolve the region about the y-axis and compute the volume of the solid thus generated.

<u>Solution</u>

Each elementary shell has volume $2\pi \cos x^2 \; x \; dx$ over the interval $[0, \frac{\sqrt{\pi}}{2}]$. So

$$V = 2\pi \int_0^{\frac{\sqrt{\pi}}{2}} x \cos x^2 dx = ... = \frac{\pi\sqrt{2}}{2}.$$

<u>Example 2</u> The region bounded by $xy = 4$, $x + y = 5$ is revolved about the x-axis. Compute the volume of the solid thus generated.

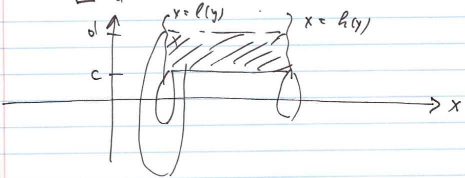


Revolve about Y axis :

$$\boxed{V = 2\pi \int_a^b x\,(f(x) - g(x))\,dx}$$

Revolve about X- axis

$$\boxed{V = 2\pi \int_c^d y\,(h(y) - \ell(y))\,dy}$$



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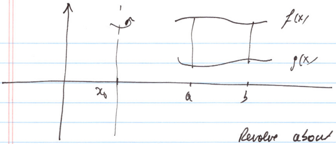
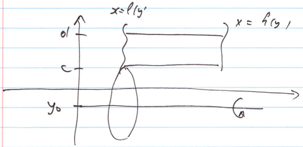


Revolve about $x = x_0$

$$\boxed{V = 2\pi \int_a^b (x - x_0)\,(f(x) - g(x))\,dx}$$

Revolving about $y = y_0$

$$\boxed{V = 2\pi \int_c^d (y - y_0)\,(h(y) - \ell(y))\,dy}$$



6.4. Length of a plane curve

Assignment: page 441, #5-8,29-32

Elementary arc length:

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

So The arc length is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Or, if x is the function of y,

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

Examples

1. Find the arc length of the curve

$$x = \frac{1}{3}(y^2 + 2)^{3/2}$$

form $y = 0$ to $y = 1$.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + y^2(y^2 + 2)} dy = \frac{4}{3}.$$

Assignment: page 447, #2,3,7,8.

1. Revolving $y = f(x)$ about the X axis:

The elementary surface area:

$$dS = 2\pi f(x) dL = 2\pi f(x)\sqrt{1 + (f'(x))^2}dx$$

Hence, the surface area of revolution is given by:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx.$$

2. Revolving $x = g(y)$ about the Y axis:

$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy.$$

Example

$y = \sqrt{4 - x^2}, \; -1 \leq x \leq 1$, revolve the curve about the X axis. Find the surface area of revolution thus generated.

6.6 Force and Work

Assignment: page 456, #7,20-24.

The work W done by a force $F(x)$ moving an object on a straight line from $x = a$ to $x = b$ is given by

$$W = \int_a^b F(x)dx.$$

Example: A force of 50 N moving an object from 0 to 10 m.

The work done is

$$\int_0^{10} 50dx = 50 \times 10 = 500Nm(or\ Joules)$$

Example: Elastic springs

Hooke's Law: The force exerted by an elastic spring is given by

$$\boxed{F(x)=kx}$$

where x is the displacement beyond the natural length of the spring and k is a constant, the spring constant.

A spring has natural length of $15in$. A $45lb$ weight stretches the spring to a length of $20in$.

Find the spring constant k.

Find the work done in stretching the spring $3in$ beyond its natural length.

Find the work done in stretching the spring from $20in$ to $25in$.

Work against gravity

Newton's Law of gravitation:

$$\boxed{F(r) = \frac{K}{r^2}}$$

where r is the distance from center of gravity and K is a constant.

So

$$W = \int_R^{R_1} \frac{K}{r^2} dr$$

is the work done by lifting an object from R to R_1.

Example

Compute the work required to lift a $1000\textit{lb}$ weight from an orbit $1000\textit{mi}$ above the earth's surface to one $2000\textit{mi}$ above.

First find the constant K.

$$1000 = \frac{K}{(4000)^2}$$

so $K = (4000)^2.1000 = 16.10^9$ in $mi^2.lb$

$$W = \int_{5000}^{6000} \frac{16.10^9}{r^2} dr$$

$$1 mi.lb = 5280 foot.lb$$

$$W = 16.10^9 \left(\frac{-1}{6000} + \frac{1}{5000} \right)$$

Work done by emptying a tank.

A rectangular (6 m by 4 m) cone shaped vat (3 m high), contains water to 2 m deep. Find the work required to pump all the water to the top of the vat. The weight density of water is $9810 N/m^3$.

Lifting a slice of water at height $x \in [0, 2]$ to the top requires an elementary work

$$dw = (3 - x)9810Adx$$

where $A(x)$ is the vat's cross sectional area at height x. This cross sectional rectangle has dimension $2r$ by $6m$ where

$$\frac{r}{x} = \frac{2}{3}$$

So $r = \frac{2x}{3}$ and $A(x) = 12 \times \frac{2x}{3} = 8x$.

$dw = (3 - x)9810 \times 8xdx$ and the work required is

$$W = \int_0^2 (3 - x)9810(8x)dx \; Nm \; (or \; joules).$$

Work done by filling a tank

Filling one layer at a time!

An elementary volume of a slice:

$$dv = A(y)dy$$

The elementary work required to lift the slice at hight y:
$dw = y\rho A(y)dy$

The total work required to fill the tank:

$$W = \int_a^b \rho y A(y)dy$$

Example

Find the work required to fill up a cylindrical tank, radius 5 ft, height 10 ft, with water from ground level.

$$W = \int_0^{10} 62.4y\pi 25dy = 25 \times 62.4\pi \times 50 lb.ft$$

6.8. Force exerted by a fluid

Assignment: page 472, #5,6,8,10.

The pressure $p = \rho h$ where h is the fluid height, ρ is the weight density of the fluid.

The elementary force exerted on either side of a submersed plate

$$dF = \rho(c - y)L(y)dy$$

where $L(y)$ is the horizontal cross sectional length of the plate at location y. The measure c is the length of fluid column.

The total force exerted on either side of the plate is

$$F = \int_a^b \rho(c - y)L(y)dy$$

Let u, v be two differentiable functions. The product rule for differentiation tells us that

$$d(uv) = udv + vdu$$

So:

$$uv = \int d(uv) = \int udv + \int vdu$$

Or, equivalently:

$$uv = \int uv'dx + \int vu'dx$$

The more useful form of this formula is

$$\int u dv = uv - \int v du$$

(Integration by Parts formula)

Examples

Compute $\int xe^{3x}dx$.

Look at this as $\int xd(\frac{1}{3}e^{3x}) = \int udv$ where $u = x$ and $v = \frac{1}{3}e^{3x}$.

Then apply the IPF:

$$\int xe^{3x}dx = \int udv = uv - \int vdu = x\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$$
$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.$$

Compute:

$$\int_1^2 x \sec^{-1} x dx$$

$$\int x \sec^{-1} x dx = \int \sec^{-1} x d(\frac{x^2}{2}) = \int udv = uv - \int vdu$$

$$= \frac{x^2}{2} \sec^{-1} x - \int \frac{dx}{x\sqrt{x^2 - 1}}$$

$$= \frac{x^2}{2} \sec^{-1} x - \frac{1}{4} \int \frac{2xdx}{\sqrt{x^2 - 1}}$$

So

$$\int_1^2 x \sec^{-1} x dx = \left(\frac{x^2}{2} \sec^{-1} x - \frac{1}{2}(x^2 - 1)^{\frac{1}{2}} \right)_1^2 = 2\frac{\pi}{3} - \frac{\sqrt{3}}{2}.$$

Application of IPF

Compute $\int \sin^{-1} x dx$

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int x d(\sin^{-1} x)$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx = x \sin^{-1} x - \sqrt{1 - x^2} + c.$$

Similarly, $\int \cos^{-1} x dx$ can be computed!

$\int \tan^{-1} x dx = x \tan^{-1} x - \int x d(\tan^{-1} x) = x \tan^{-1} x - \int \frac{x}{1+x^2} dx =$
$x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.$

$$\sin^n x dx = \sin^{n-1} x \sin x = \sin^{n-1} x d(-\cos x)$$

So

$$\begin{aligned}
\int \sin^n x dx &= \int \sin^{n-1} x d(-\cos x) = \int u dv \\
&= uv - \int v du \\
&= -sin^{n-1} x \cos x + \int \cos x (n-1) \sin^{n-2} x \cos x dx \\
&= -\cos x \sin^{n-1} x + (n-1) \int (1 - sin^2 x) \sin^{n-2} x dx \\
&= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx \\
& \quad -(n-1) \int \sin^n x dx
\end{aligned}$$

Therefore

$$(1+n-1)\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

Or

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

In the same manner

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Example:

Compute $\int_0^{\frac{\pi}{4}} \sin^4 dx$

$$\int \sin^4 x dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x dx$$

$$= -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}[(-)\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx]$$

$$= -\frac{1}{4}\cos x \sin^3 x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$$

and

$$\int_0^{\frac{\pi}{4}} \sin^4 x dx = -\frac{1}{4} + \frac{3\pi}{32}.$$

Integrals of powers of sine and cosine

$$\int \sin^m x \cos^n x dx$$

- If both exponents are even, then using the identity $\sin^2 x + \cos^2 x = 1$ reduces the integrand to a sum of even powers of $\cos x$ or $\sin x$.
- If not, the integrand can be reduced to sums of one of the following:

$$\sin^m x \cos x$$

or

$$\cos^n x \sin x$$

Then use the substitution method to compute the integral.

examples

$$\int \sin^{2k} x \cos^{2l} x dx = \int \sin^{2k} x (1 - \sin^2 x)^l dx$$

$$
\begin{aligned}
\int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x \cos^{2k} x \cos x dx \\
&= \int \sin^m x (1-\sin^2 x)^k \cos x dx.
\end{aligned}
$$

Making the substitution $u = \sin x$, the integral becomes:

$$\int u^m (1-u^2)^k du$$

Similarly, with the substitution $u = \cos x$, the integral

$$\int \sin^{2l+1} x \cos^n x dx$$

becomes

$$-\int (1-u^2)^l u^n du$$

Integration of powers of secant and tangent

$$\int \tan^m x \sec^n x dx$$

- If both m and n are odd, then the substitution $u = \sec x$ will work.
- If at least one of m and n is even, then use the identity $\sec^2 x = 1 + \tan^2 x$ to change into powers of $\sec x$ or $\tan x$ only.

Examples

$$\int \tan^{2k+1} x \sec^{2l+1} x dx = \int (\sec^2 x - 1)^k \sec^{2l} \tan x \sec x dx$$
$$= \int (u^2 - 1)^k u^{2l} du$$

where $u = \sec x$.

- $\int \tan^m x \sec^{2l} x dx = \int \tan^m x (1 + \tan^2 x)^l dx$
- $\int \tan^{2k} x \sec^n x dx = \int (\sec^2 x - 1)^k \sec^n x dx$

$$
\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
&= \int \frac{du}{u} \quad \textit{where } u = \sec x + \tan x \\
&= \ln|\sec x + \tan x| + c
\end{aligned}
$$

$$\begin{aligned}
\int \sec^n x dx &= \int \sec^{n-2} x \sec^2 x dx \\
&= \int \sec^{n-2} x d(\tan x) \textit{ Integration by parts follows} \\
&= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan^2 x dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx \\
(1+n-2) \int \sec^n x dx &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x dx \\
\int \sec^n x dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx
\end{aligned}$$

$$
\begin{aligned}
\int \tan x dx &= \int \frac{u}{1+u^2} du \quad \textit{where } u = \tan x \\
&= \frac{1}{2} \int \frac{dv}{v} \quad \textit{where } v = 1+u^2 \\
&= \frac{1}{2} \ln|1+\tan^2 x| + c \\
&= \frac{1}{2} \ln|\sec^2 x| + c \\
\int \tan x dx &= \ln|\sec x| + c
\end{aligned}
$$

$$
\begin{aligned}
\int \tan^m x dx &= \int \tan^{m-2} x \tan^2 x dx \\
&= \int \tan^{m-2} x (\sec^2 x - 1) dx \\
&= \int \tan^{m-2} x \sec^2 x dx - \int \tan^{m-2} dx \\
\int \tan^m x dx &= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x dx \quad \textit{for } m \neq 1.
\end{aligned}
$$

Examples

Compute

$$\int \tan^4\theta \sec^4\theta d\theta$$

$$\int x \tan^2(x^2) \sec^2(x^2) dx$$

$$\int_0^{\frac{\pi}{6}} \sec^3\theta \tan\theta d\theta$$