Lecture 3: Mathematical Foundation of Quantum Mechanics and its Interface with Classical Mechanics

February 13, 2015

1 Some "non"-Classical Features of Quantum Mechanics

- Discreteness of some Observables: Energy, Angular Momentum
- Superposition Effect and Wave-Particle Duality
- Indeterminacy

2 Complex Vector Space

We now discuss the abstract formulation of the complex vector space, which can be considered as a generalization of the 3d space vector case:

1. We introduce a complex vector $\mid \psi \rangle$

2. All possible $\mid \psi \rangle$ together with set of rules on these vectors define the complex vector space.

3. The space also has a complex scalar α.

4. Rules: Vector Additions and multiplication by scalar:

The product of a vector and scalar as well as sum of the vectors multiplied by ant scalars produce another vector in the same space.

$$\mid \psi \rangle = \alpha \mid \psi_1 \rangle + \beta \mid \psi_2 \rangle \tag{1}$$

where $\mid \psi \rangle$ is a vector in the same space.

The above procedure satisfies the following rules:

a) *Distributive Rule:*

$$\begin{aligned} (\alpha + \beta) \mid \psi \rangle &= \alpha \mid \psi \rangle + \beta \mid \psi \rangle \\ \alpha(\mid \psi_1 \rangle + \mid \psi_1 \rangle) &= \alpha \mid \psi_1 \rangle + \alpha \mid \psi_2 \rangle \end{aligned} \tag{2}$$

a) *Associative Rule:*

$$
\begin{aligned}
(\mid \psi_1\rangle + \mid \psi_2\rangle) + \mid \psi_3\rangle &= \mid \psi_1\rangle + (\mid \psi_2\rangle + \mid \psi_3\rangle) \\
\alpha(\beta \mid \psi\rangle) &= (\alpha\beta) \mid \psi\rangle
\end{aligned} \tag{3}
$$

5.The space has a *zero* vector $\mid 0\rangle \equiv 0$ defined as:

$$\mid \psi\rangle + 0 = \mid \psi\rangle \tag{4}$$

6. One can define a basis vectors $\mid \psi_n\rangle$ such that

$$\mid \psi\rangle = \sum_n \alpha_n \mid \psi_n\rangle \tag{5}$$

where α_n's are "components"

7. *Scalar Product:* We define the scalar product in the vector space

$$(\mid \phi\rangle, \mid \psi \mid) \tag{6}$$

such that

$$(\mid \phi\rangle, \mid \psi \mid) = (\mid \psi\rangle, \mid \phi \mid)^* \tag{7}$$

which ensures that $(\mid \psi\rangle, \mid \psi \mid)$ is a real scalar.

For further discussion we use Dirac's notation in such a way that scalar product of Eq.(6) will be defined as

$$\langle\phi \mid \psi\rangle = \langle\psi \mid \phi\rangle^\dagger \tag{8}$$

$\boxed{\text{The scalar product is linear in } |\rangle \text{ part and antilinear in } \langle| \text{ part.}}$

That is if $\mid \psi\rangle = \alpha \mid \psi_1\rangle + \beta \mid \psi_2\rangle$ then

$$\langle\phi \mid \psi\rangle = \alpha\langle\phi \mid \psi_1\rangle + \beta\langle\phi \mid \psi_2\rangle \tag{9}$$

and

$$\langle\psi \mid \phi\rangle = \alpha^*\langle\psi_1 \mid \phi\rangle + \beta^*\langle\psi_2 \mid \phi\rangle \tag{10}$$

8. *Inner product:* Requiring to be

$$\boxed{\text{Positive Definitive}}$$
$$\boxed{\langle\psi \mid \psi\rangle \geq 0}$$

that is $\langle\psi \mid \psi\rangle = 0$ only when $\mid \psi\rangle = 0$.

Such condition allow to define the a *Norm*

$$\mid\mid \psi\rangle \mid = \sqrt{\langle\psi \mid \psi\rangle} \tag{11}$$

9. *Orthogonality* of vectors: Two state vectors $\mid \phi\rangle$ and $\mid \psi\rangle$ are orthogonal if

$$\langle\phi \mid \psi\rangle = \langle\psi \mid \phi\rangle = 0 \tag{12}$$

This property defines the basis vectors:

$$\langle\psi_m \mid \psi_n\rangle = 0 \quad \text{if} \quad m \neq n \tag{13}$$

One can further restrict the definition of basis vector requiring:

$$\langle\psi_m \mid \psi_n\rangle = \delta_{mn} \tag{14}$$

The above condition defines so called *orthonormal* basis vectors.

10. *Component Representation*

Using above defined orthonormality condition one can show that any state vector $\mid \psi\rangle$ can be expanded through the sum of the basis vectors as follows:

$$\mid \psi\rangle = \sum_n \alpha_n \mid \psi_n\rangle, \tag{15}$$

where n defines the "dimensionality" of the vector space. Using orthonormality condition of Eq.(14) one obtains:

$$\alpha_n = \langle\psi_n \mid \psi\rangle \tag{16}$$

11. *Completeness of the space* Using Eqs.(15) and (16) we obtain

$$\langle\psi \mid \psi\rangle = \sum_n \alpha_n\langle\psi \mid \psi_n\rangle = \sum_n \mid \alpha_n \mid^2 \tag{17}$$

where n can be infinity.

The vector space is called em complete if additional condition is imposed such that $\langle\psi \mid \psi\rangle$ is finite, i.e.

$$\sum_n \mid \alpha_n \mid^2 \leq \infty \tag{18}$$

The complex vector space which satisfies the conditions of *positive definiteness of the inner product* **8.** and completeness **11.** is called $\boxed{\text{Hilbert Space}}$.

3 Linear Operators

Linear operators $\hat{A}$ are functions defined in the complex vectors space with following properties.

1. Operator's action on a vector results in another vector in the same vector space:

$$\hat{A} \mid \psi\rangle = \mid \phi\rangle, \tag{19}$$

where $| \phi\rangle$ is another vector in the same complex space.

2. Linearity of Operators: Operator $\hat{A}$ is linear if:

$$\hat{A}(\alpha \mid \psi_1\rangle + \beta \mid \psi_2\rangle) = \alpha\hat{A} \mid \psi_1\rangle + \beta\hat{A} \mid \psi_2\rangle) \tag{20}$$

3. Matrix Elements of Operators: If the basis vectors are defined in the given vector space, then one can define the matrix element as follows: First act by $\hat{A}$ on one of the basis vector which will produce some other vector in the vector space which itself can be expressed through the basis vectors according to Eq.(15):

$$\hat{|}\psi_n\rangle = | \phi\rangle = \sum_k A_{kn} \mid \psi_k\rangle, \tag{21}$$

where A_{kn} are the components of the expansion of $| \phi\rangle$ by basis vectors.

One can find A_{kn}'s by multiplying the left part of Eq.(21) by $\langle\psi_m \mid$ and using the orthonormalilty relation of Eq.(14). One obtains:

$$A_{mn} = \langle\psi_m \mid \hat{A} \mid \psi_n\rangle \tag{22}$$

where A_{mn}'s are called *matrix elements* of operator $\hat{A}$ in the *basis* of $| \psi_n\rangle$. Clearly for given operator it is dependent on the choice of the basis vectors $| \psi_n\rangle$.

4. Usefulness of Matrix Elements. If the matrix elements of the given operator is known in some given basis, then one can calculate the action of this operator on any vector $| \psi\rangle$. For example

$$\hat{A} \mid \psi\rangle = \sum_n \hat{A}\alpha_n \mid \psi_n\rangle = \sum_{mn} A_{mn}\alpha_n \mid \psi_m\rangle \tag{23}$$

where in the second step we used Eq.(15) to expand the sate $| \psi\rangle$ through the basis ψ_n with components α_n.

5. Some Rules of Engagement

- **Sum of operators:**

$$[\hat{A} + \hat{B}] \mid \psi\rangle = \hat{A} \mid \psi\rangle + \hat{B} \mid \psi\rangle \tag{24}$$

- **Product of scalar and an operator:**

$$(\alpha\hat{A}) \mid \psi\rangle = \alpha\hat{A} \mid \psi\rangle \tag{25}$$

- **Product of operators:**

$$\hat{A}\hat{B} \mid \psi\rangle = \hat{A}(\hat{B} \mid \psi\rangle) \tag{26}$$

Operators are called *commutative* if:

$$\hat{A}\hat{B} \mid \psi\rangle = \hat{B}\hat{A} \mid \psi\rangle \tag{27}$$

and *noncummutative*

$$\hat{A}\hat{B} \mid \psi\rangle \neq \hat{B}\hat{A} \mid \psi\rangle \tag{28}$$

In this case we write that $[\hat{A}, \hat{B}] \neq 0$. *Hereafter whenever we discuss the product of operators say* $\hat{A}, \hat{B}$ *it is implicit that the product acts on the state vector at its left.*

- **Matrix element of the product of operators:** Since product of two operators is another operator:

$$\hat{C} = \hat{A}\hat{B} \tag{29}$$

one should be able to express the matrix element of the $\hat{C}$ through the matrix elements of $\hat{A}$ and $\hat{B}$

$$C_{mn} = \sum_k A_{mk} B_{kn} \tag{30}$$

Example of an Operator: One simple example of operators constructed from vectors is:

$$\hat{A} \equiv \mid \phi_1\rangle\langle\phi_2 \mid \tag{31}$$

One can calculate the action of this operator on any state vector using the definition of the scalar product of Eq.(8).

$$\hat{A} \mid \psi >= \mid \phi_1\rangle\langle\phi_2 \mid \psi\rangle \tag{32}$$

where $\langle\phi_2 \mid \psi\rangle$ is a scalar product (number). Thus right hand side is another vector in the same vector space and condition of Eq.(65) is fulfilled.

6. The Identity Operator: The identity operator $\hat{I}$ is defined as:

$$\hat{I} \mid \psi\rangle = \mid \psi\rangle \tag{33}$$

Example of the Identity Operator: Using an orthonormal basis vectors $\mid \psi_n\rangle$ one can construct an identity operator as follows:

$$\hat{I} = \sum_n \mid \psi_n\rangle\langle\psi_n \mid \tag{34}$$

It is easy to show now that for any arbitrary $\mid \psi_n\rangle$

$$\hat{I} \mid \psi\rangle = \sum_n \mid \psi_n\rangle\langle\psi_n \mid \psi\rangle = \sum_n \alpha_n \mid \psi_n\rangle = \mid \psi\rangle, \tag{35}$$

where $\alpha_n = \langle\psi_n \mid \psi\rangle$.

7. Inverse of an Operator: The operator $\hat{A}^{-1}$ is called an inverse of the operator $\hat{A}$ if

$$\hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = \hat{I} \tag{36}$$

If operator $\hat{A}$ does not have its inverse one call such operator as a singular operator. For finite dimensional space such situation corresponds to the $det[A] = 0$, where A is the matrix with the matrix element of operator $\hat{A}$ calculated with any orthonormal basis.

It can be shown that

$$(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1} \tag{37}$$

8. Eigenvectors and Eigenvalues:

If for given operator $\hat{A}$ there is such a vector state that

$$\hat{A} \mid \psi\rangle = \lambda \mid \psi\rangle \tag{38}$$

where λ is a scalar, we call it an eigenvalue and the state $\mid \psi\rangle$ *eigenstate*.

For finite dimension and for any orthornormal basis the problem of finding eigenvalues is reduced to finding the solution of equation

$$det(A - \lambda) = 0 \tag{39}$$

where A is the matrix composed of the matrix elements of the operator $\hat{A}$.

9. Hermitean Operator:

If for all vectors of $\mid \phi\rangle$ and $\mid \psi\rangle$ to operators $\hat{B}$ and $\hat{A}$ are related through the reflation

$$\langle\phi \mid \hat{B} \mid \psi\rangle = \langle\psi \mid \hat{A} \mid \phi\rangle^* \tag{40}$$

$\hat{B}$ is called the *Hermitean Conjugate or adding* of operator $\hat{A}$ and it is written as:

$$\hat{B} = \hat{A}^\dagger \quad \text{or} \quad B^\dagger = \hat{A}. \tag{41}$$

It can be shown that the matrix element of $\hat{B}$ is related to the one of $\hat{A}$ through the relations:

$$B_{mn} = (A^T_{mn})^* = A^*_{nm} \tag{42}$$

where ^{T} means transparent of the matrix.

Some properties:

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{43}$$

The operator $\hat{A}$ is called *Hermitean or Self-Adjoint* if

$$A^\dagger = A \tag{44}$$

in this case the matrix elements of such operator will have certain symmetry:

$$A_{mn} = A^*_{nm} \tag{45}$$

Some properties of Hermitean operators:

- The eigenvalues of a Hermitean operator is real
- The opposite is also true: if all eigenvalues are real then operator is Hermitean
- The eigenvectors of a Hermitean operator with different eigenvalues are orthogonal
- The eigenvectors of Hermitean operator can be chosen to be a complete orhonormal basis.
- The matrix consisting of the matrix elements of Hermitean operator in the basis of its own eigenvectors is *diagonal*.

10. Commutation of Operators:
If two different operators have same eigenvalues then they commute:

$$[\hat{A}\hat{B}] = 0 \tag{46}$$

The opposite is also true: *If two operators do not commute they can not have same eigenstates.*
11. Unitary Operators:
Let us consider operator $\hat{U}$ with the following property:

$$\mid \phi_1\rangle = \hat{U} \mid \psi_1\rangle \quad \text{and} \quad \mid \phi_2\rangle = \hat{U} \mid \psi_2\rangle \tag{47}$$

such that

$$\langle \phi_1 \mid \phi_2\rangle = \langle \psi_1 \mid \psi_1\rangle. \tag{48}$$

Such an operators called *Unitary* operator.

To see what particular property the above relation ascribes to the operator $\hat{U}$, we first show that

$$\langle \phi_1 \mid = \langle \psi_1 \mid U^\dagger. \tag{49}$$

To see these we recall the definition of the scalar product of Eq.(**??**) and Hermithen conjugation of Eq.(40). According to Eq.(40) for any given operator $\hat{A}$

$$\langle \psi_1 \mid \hat{U}^\dagger \mid \phi_3\rangle = (\langle \phi_3 \mid \hat{U} \mid \psi_1\rangle)^* = (\langle \phi_3 \mid \phi_1\rangle)^* \tag{50}$$

and then according to Eq.(8)

$$(\langle \phi_3 \mid \phi_1\rangle)^* = \langle \psi_1 \mid \phi_3\rangle \tag{51}$$

Comparison of the RHS of Eq.(51) with the LHS part of Eq.(50) shows the validity of the relation of Eq.(49).

Using now Eq.(49) in (52) one obtains:

$$\langle \phi_1 \mid \phi_2\rangle = \langle \psi_1 \mid \hat{U}^\dagger U \mid \psi_1\rangle = \langle \psi_1 \mid \psi_1\rangle, \tag{52}$$

from which it follows that:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = I \tag{53}$$

exercise: show that the middle part of the above equation is indeed true.

From Eq.(53) it follows that

$$\hat{U}^\dagger = U^{-1}. \tag{54}$$

This represents the *main* property of Unitary operators.

12. Hermitean Exponents of Unitary Operators:

If unitary operator can be expressed in the form of exponent:

$$U = e^{-iG} \tag{55}$$

then one can show that the operator G is an hermitean operator.

To prove this statement one first observes that the exponential representation of the operator is understood in the form of the power series:

$$U = \sum_{n=0}^{\infty} \frac{1}{n!}(-i)^n G^n. \tag{56}$$

Using this representation and the condition of the unitarity one obtains

$$U^\dagger U = e^{-i(G-G^\dagger)} = 1 \tag{57}$$

from which it follows that $G = G^\dagger$, that is G is hermitean.

4 Relating to the Reality

One of the most unique property of quantum physics is the *concept of the measurement.* While in Classical Mechanics one in principle can isolate a single particle and make the measurement, in Quantum Mechanics we inherently deal with many particle like atoms and there is not way we can isolate single atom and make a measurement. Thus inevitability we deal with the essemble of particles or multitude of measurements of the quantity we are interested in.

This brings us to the concept of the *Statistical Measurement"* in which we need to average by large number of singe measurements. Such a statistical measurement of the magnitude of the given observables is related to the statistical expectation value of the observable, (A_{ev}), according to which one measures the "value" of the observable which represents a sum of all possible values of observable, (A_n) weighted by their probability of the outcome, (P_n):

$$A_{EV} = \sum_n P_n A_n. \tag{58}$$

As we will see below, this relation will serve us as an "human" (or classical) interface to the quantum world.

4.1 Defining Quantum Reality

To construct the above discussed "interface" we introduce set of **Axioms**

(I). All physical states are represented by complex vector $\mid \psi\rangle$ in the Hilbert space that posesses certain property ξ.

(II). If the system is in ϕ - the probability that it exhibits the property ξ is

$$\mid \langle\psi \mid \phi\rangle \mid^2 \tag{59}$$

Accordingly we call $\langle\psi \mid \phi\rangle$ as $\boxed{\textit{Probability Amplitude}}$.

(III). We introduce an observable A which is characterized with the list of measurable values, this list can be discrete (finite or infinite) or continuous (will define later). At the moment we assume that the list is discrete with the measurable values λ_i.

(IV). There is always a "collapsed" state $\mid \lambda_i\rangle$ such that if observable A is measured at this state it certainly gives the magnitude of λ_i but no other magnitudes of λ_j with $j \neq i$. The above statement using the Axiom II can be presented mathematically as follows:

$$\langle\lambda_i \mid \lambda_j\rangle = \langle\lambda_j \mid \lambda_i\rangle = \delta_{ij} \tag{60}$$

We now develop from these four-axioms observing the "utter" similarity between Eq.(60) and (**??**). This tells us that the "collapsed" states can be chosen as a **basis** states.

With such a choice we now can proceed observing:

1. Projection Theorem: Any state $\mid \psi$ can be expressed through the collapsed states according to Eq.(15):

$$\mid \psi\rangle = \sum_n \alpha_n \mid \lambda_n\rangle, \tag{61}$$

where α_n's are defined according to Eq.(16)

$$\alpha_n = \langle\lambda_n \mid \psi\rangle \tag{62}$$

and $\mid \alpha_n \mid^2$ defines the probability that measurements of the observable A in the state $\mid \psi\rangle$ yields λ_i.

2. Closure Theorem: Since any measurement at the $\mid \psi\rangle$ state will result some value of λ_i and since λ_i is cover all possible magnitudes of the observable A, then

$$\sum_n \mid \alpha_n \mid^2 = 1 = \sum_n \mid \langle\lambda_n \mid \psi\rangle \mid^2 = \sum_n \langle\psi \mid \lambda_n\rangle\langle\lambda_n \mid \psi\rangle. \tag{63}$$

Using now the relation of Eq.(34) in the RHS part of the above equation we arrived at:

$$\langle\psi \mid \psi\rangle = 1 \tag{64}$$

Which states that any state ψ at which the observable A can be measured with the discrete outcomes of λ_i has a inner product of unity (or is normalized to one).

3. Operator Theorem: For any observable A one can construct an operator for which the states $\mid \lambda_i\rangle$ are eigenstates and magnitudes λ_i are eigenvalues.

Such an operator can be constructed as follows:

$$\hat{A} = \sum_n \lambda_n \mid \lambda_n\rangle\langle\lambda_n \mid \tag{65}$$

and one can see that

$$\hat{A} \mid \lambda_i\rangle = \sum_n \lambda_n \mid \lambda_n\rangle\langle\lambda_n \mid \lambda_i\rangle = \lambda_i \mid \lambda_i, \tag{66}$$

where in the RHS part we used relation: $\langle\lambda_n \mid \lambda_i\rangle = \delta_{ni}$.

As it follows from Eq.(65) if eigenvalues λ_i are real then operator will be Hermitean $\hat{A}^\dagger = \hat{A}$.

4. Theorem on Multiple Observables: According to the rule of the operator commutation of Eq.(46) if the quantum system has more than one observables that can be measured simultaneously then their corresponding operators will commute. Also the opposite is true that if two operators are not commuting one can not find a state at which corresponding observables can be *eigenestates*.

4. Interface Theorem: If one chooses set of eigenstates λ_b of the given observables A as a basis state with corresponding eigenvalues λ_n, then using Eq.(59) we can define quantum-mechanics expectation value of the operator $\hat{A}$ for any arbitrary state $\mid \psi\rangle$ as follows:

$$\langle\psi \mid \hat{A} \mid \psi\rangle = \sum_n \lambda_n \langle\psi \mid \lambda_n\rangle\langle\lambda_n \mid \psi\rangle = \sum_n \lambda_n \mid \langle\lambda_n \mid \psi\rangle \mid^2 \tag{67}$$

where in the second term we used the relation of Eq.(65). One can recognize in this equation the definition of the expectation value (58) if we treat the $\mid \langle\lambda_n \mid \psi\rangle \mid^2$ as a probability according to Eq.(59). This is the *interface* equation that relates the quantum-mechanical expectation value to the experimental expectation value that is measured in the experiment dealing with the large number of the assemble of quantum objects like atoms.

5 Continuous Eigenvalues

So far we discussed an observables that can have discrete values and used the Hilbert space with state vectors with discrete number of components and operators having discrete eigenvalues.

However we will need also consider the quantum analogy of continuous observables, for which we need to generalize the above discussed framework for continuous eigenvalues.

5.1 Dirac's Delta Function

One of the important steps in formulating the mathematics for continuous vector space is the generalization of the Kronecker delta function δ_{ij}, which was introduce by Dirac in 1930's with two main properties:

$$\delta(x) = 0 \quad \text{when} \quad (x \neq 0) \tag{68}$$
$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Based on these two properties one can show the following relations:

$$\int_a^b \delta(x)dx = \begin{cases} 1 & \text{if } a < x < b; \\ 0 & \text{otherwise.} \end{cases} \tag{69}$$

For any function $f(x)$

$$\int_a^b f(x)\delta(x-c)dx = f(c) \quad \text{if} \quad a < c < b \tag{70}$$

Also, several following relations

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{71}$$

$$\int_a^b f(x)\delta(F(x) - F(c))dx = \frac{f(c)}{\mid F'(c) \mid} \quad \text{if} \quad a < c < b \tag{72}$$

The useful integral representation of the δ function is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}dk \tag{73}$$

The delta function can be generalized for 3-dimensional space with the definition:

$$\delta^3(r) = \delta(x)\delta(y)\delta(z) \tag{74}$$

5.2 Continuous Observables

For an example of continuous observable we will consider the 1-dimensional coordinate x. We assume that there are collapsed state of coordinate $\mid x\rangle$ with observed position of x. In this case according to *Operator Theorem*, we always can construct operator $\hat{x}$ (similar to Eq.(65) such that

$$\hat{x} \mid x\rangle = x \mid x\rangle. \tag{75}$$

Then these states $| x\rangle$ can be used as a basis states with orthonormality condition generalized from Eq.(60) for continuous observables as:

$$\langle x' \mid x\rangle = \delta(x - x') \tag{76}$$

Now we can generalized the *Projection Theorem* for the continuous case expressing given state vector $| \Psi\rangle$ through the basis states $| x\rangle$ in the form

$$| \psi\rangle = \int\limits_{-\infty}^{\infty} \alpha(x) \mid x\rangle dx \tag{77}$$

where "components" of $| \psi\rangle$ along the basis states are

$$\alpha(x) = \langle x \mid \psi\rangle \tag{78}$$

where

$$\rho(x) = |\alpha(x)|^2 \tag{79}$$

is called probability density and the following integral

$$\int\limits_{x_0}^{x_0+\Delta} \rho(x)dx \tag{80}$$

gives the probability that one will find the position of the quantum state $| \psi\rangle$ between x_0 and $x_0 + \Delta$.

Identity Operator: One can generalize the construction of identity operator in Eq.(34) in the form of

$$\hat{I} = \int\limits_{-\infty}^{\infty} | x\rangle\langle x \mid dx. \tag{81}$$

Using Eqs.(77) and (78) one obtains

$$\hat{I} \mid \psi\rangle = \int\limits_{-\infty}^{\infty} | x\rangle\langle x \mid \psi > dx = \int\limits_{-\infty}^{\infty} \alpha(x) \mid x\rangle dx =| \psi\rangle \tag{82}$$

Thee-Dimensional Case: It is rather straightforward to generalize the above discussed one-dimensional case to 3d case. For this one needs to introduce the collapse states of $| r\rangle >$ which is collapsed to the position characterized by coordinates x, y and z. The $\hat{r}$ operator will have a property of

$$\hat{r} \mid r\rangle = \vec{r} \mid r\rangle \tag{83}$$

and the probability density will be defined as

$$\rho(r) =| \langle r \mid \psi\rangle |^2 \tag{84}$$

6 Correspondence Principle

The last step in relating the abstract algebra of the linear operators in the complex vector state to the classical dynamic is the *Correspondence Principle* or *Ehrenfest's Principle*. It defines the minimal numb paradigms that we need to accept to develop the quantum dynamics and relate to to classical comprehension of it.

The starting point of the Correspondence principle is that the *interface* of the Quantum and Classical mechanics is the expectation value of the Hermitean operator $\langle\mid \hat{A} \mid\rangle$ which is associated with the classical measurements.

Since in such measurements one measures all known classical observables then one can state that

(I)) The list of the observables in Classical Mechanics should be included in the list of Quantum Mechanics *observables.*

Thus for any known classical observable O_{cl} in Quantum Mechanics there will be collapsed state $\mid O_n\rangle$ with such a quantum observables O_n for which according to Operator Theorem of Eq.(65) one can construct the Hermitean operator $\hat{O}$ for which

$$\hat{O} \mid O_n\rangle = O_n \mid O_n\rangle \tag{85}$$

(II) The Symmetries of Classical Systems should hold in Quantum Systems too.

The operational realization of the above statement is that the transformation operators related to the given symmetry (as it was discussed in Lecture 1) no acts not on momentum (velocity) and coordiante variables as in classical mechanics but on most fundamental of the quantum-mechanics reality the state vector $\mid \psi\rangle$.

So, the given symmetry transformation $\hat{S}$, instead of considering the operation:

$$\hat{S} : r_i = r_i' \quad \text{and} \quad \hat{S} : p_i = p_i' \tag{86}$$

one needs to consider

$$\hat{S} \mid \psi\rangle = \mid \psi'\rangle \tag{87}$$

It is worth mentioning that the opposite is not true there are symmetries in Quantum Mechanics which do not propagate to Classical domain, such as spin-symmetry, particle-antiparticle symmetry, color charge symmetry, etc

(III) The time dependence of the expectation values of any operator corresponding to the quantum observable, $\langle\mid \hat{O} \mid\rangle$ should follow the time dependence of the corresponding classical observable.

This principle completes the all necessary conditions for establishing classical "comprehension" of quantum system. The mathematical consequence of it is that the time dependence of $\langle\mid \hat{O} \mid\rangle$ should be defined by classical Poission brackets (discussed in Lecture 1 Eqs.(85) and (86))

$$\frac{d\langle\mid \hat{O} \mid\rangle}{dt} = \frac{\partial\langle\mid \hat{O} \mid\rangle}{\partial t} + \left\{\langle\mid \hat{O} \mid\rangle, H\right\} \tag{88}$$

7 Appendix

7.1 If Two Hermitean Operators Commute They Have Same Eigenestate

We would like to prove this theorem using matrix approach.

Let $| \psi_n >$ to be eigenstate of operator $\hat{A}$, with eigenvalue a_n, i.e.

$$\hat{A} \mid \psi_n\rangle = a_n \mid \psi_n\rangle \tag{89}$$

Using this for the matrix elements of $\hat{A}$ we obtain:

$$\langle\psi_m \mid \hat{A} \mid \psi_n\rangle = a_n\delta_{mn} \tag{90}$$

Now we consider the other operator $\hat{B}$ that commutes with $\hat{A}$, i.e.

$$\hat{A}\hat{B} - \hat{B}\hat{A} = 0 \tag{91}$$

One can now calculate the matrix elements of the product of $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ operators using the rules of Eq.(30).

$$\langle\psi_m \mid \hat{A}\hat{B} \mid \psi_n\rangle = \sum_k A_{mk}B_{kn} = \sum_k a_m\delta_{mk}B_{kn} = a_m B_{mn} \tag{92}$$

and

$$\langle\psi_m \mid \hat{B}\hat{A} \mid \psi_n\rangle = \sum_k B_{mk}A_{kn} = \sum_k B_{mk}a_n\delta_{nk} = a_n B_{mn} \tag{93}$$

Subtracting above two equations and using condition of Eq.(91) one obtains

$$(a_m - a_n)B_{mn} = 0 \tag{94}$$

which means $B_{mn} = 0$ if $m \neq n$ therefore the matrix elements of B_{mn} in the basis of the eigenstates of operator $\hat{A}$, $\mid \psi\rangle$ are diagonal. But according to above discussion of the properties of Hermitean operators, the matrix element of the operator is diagonal in the basis of its eigenstates. Therefore one concludes that ψ_n is also an eigenstate for operator $\hat{B}$.