MAC 2313 (Calculus III) — Answers
Test 1, September 21, 2016

Name: **PID:**

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Always do your best. Total=85 points on 3 pages.

1 [10] Describe the given surface according to the values of the parameter m; if it is a sphere, state its radius and center. If it is a point, state its coordinates. $x^2+y^2+z^2-4x-6my+10z+38=0$.

$$x^2-4x+4+y^2-6my+9m^2+z^2+10z+25=-38+4+9m^2+25$$
$$=-9+9m^2$$
$$=9(m^2-1) \text{ or}$$
$$(x-2)^2+(y-3m)^2+(z+5)^2=9(m^2-1)$$

sign of m^2-1: $-\infty$ $+$ -1 $-$ 1 $+$ $+\infty$

- If $m<-1$ or $m>1$; sphere, Center $=(2,3m,-5)$, radius $=3\sqrt{m^2-1}$
- If $m=-1$ or $m=1$; point, $(2,-3,-5)$ and $(2,3,-5)$ respectively.
- If $-1<m<1$; no graph

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2. [10] a) Set $\vec{u}=\vec{i}-4\vec{j}+2\vec{k}$, $\vec{v}=2\vec{i}+\vec{j}+\vec{k}$ and $\vec{z}=-2\vec{i}+\vec{j}+3\vec{k}$. a) Show that $\vec{u}$, $\vec{v}$ and $\vec{z}$ are pairwise orthogonal vectors. b) Let $\vec{w}=3\vec{i}+2\vec{j}-4\vec{k}$. Find three scalars a, b and c such that $\vec{w}=a\vec{u}+b\vec{v}+c\vec{z}$.

a) $\vec{u}\cdot\vec{v}=1(2)-4(1)+2(1)=0$, $\vec{u}\cdot\vec{z}=1(-2)-4+2(3)=0$
$\vec{v}\cdot\vec{z}=2(-2)+1(1)+1(3)=0$; so $\vec{u},\vec{v},\vec{z}$ are pairwise orthogonal.

b) $\vec{w}\cdot\vec{u}=a\vec{u}\cdot\vec{u}=a\|\vec{u}\|^2$; so $a=\dfrac{\vec{w}\cdot\vec{u}}{\|\vec{u}\|^2}=\dfrac{3(1)+2(-4)-4(2)}{1+16+4}=-\dfrac{13}{21}$

$\vec{w}\cdot\vec{v}=b\vec{v}\cdot\vec{v}=b\|\vec{v}\|^2$; so $b=\dfrac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|^2}=\dfrac{3(2)+2(1)-4(1)}{4+1+1}=\dfrac{4}{6}=\dfrac{2}{3}$

$\vec{w}\cdot\vec{z}=c\,\vec{z}\cdot\vec{z}=c\|\vec{z}\|^2$; so $c=\dfrac{\vec{w}\cdot\vec{z}}{\|\vec{z}\|^2}=\dfrac{3(-2)+2(1)-4(3)}{4+1+9}=-\dfrac{16}{14}=-\dfrac{8}{7}$

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3. [14] Let $\vec{q}=\vec{i}-\vec{j}+4\vec{k}$, and $\vec{r}=-2\vec{i}+\vec{j}-\vec{k}$. a) Find the vector component of $\vec{q}$ that is orthogonal to $\vec{r}$.

The required vector is $\vec{q}-\mathrm{proj}_{\vec{r}}(\vec{q})$. Now, $\mathrm{proj}_{\vec{r}}(\vec{q})=\dfrac{\vec{q}\cdot\vec{r}}{\|\vec{r}\|^2}\vec{r}=\dfrac{-2-1-4}{4+1+1}\vec{r}=-\dfrac{7}{6}\vec{r}$

$\vec{q}-\mathrm{proj}_{\vec{r}}(\vec{q})=\langle 1,-1,4\rangle+\frac{7}{6}\langle -2,1,-1\rangle$
$=\left\langle \frac{6-14}{6}, -\frac{6+7}{6}, \frac{24-7}{6}\right\rangle=\left\langle -\frac{4}{3}, \frac{1}{6}, \frac{17}{6}\right\rangle$

b) If θ is the angle between $\vec{r}$ and $\vec{q}$, find $\cos(\theta)$ and $\sin(\theta)$.

$\cos\theta=\dfrac{\vec{q}\cdot\vec{r}}{\|\vec{q}\|\|\vec{r}\|}=\dfrac{-7/\sqrt{6}}{\sqrt{18}}=-\dfrac{7}{6\sqrt{3}}$

$\sin\theta=\dfrac{\|\vec{q}\times\vec{r}\|}{\|\vec{r}\|\|\vec{q}\|}=\dfrac{\sqrt{9+49+1}}{\sqrt{6}\sqrt{18}}=\dfrac{\sqrt{59}}{6\sqrt{3}}$

$$\vec{q}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 4\\ -2 & 1 & -1\end{vmatrix}=(1-4)\vec{i}-(-1+8)\vec{j}+(1-2)\vec{k}=-3\vec{i}-7\vec{j}-\vec{k}.$$

c) If a force $\vec{F}=-2\vec{q}$ is applied to move an object 4 meters in the direction of the vector $\vec{r}$, find the work done by $\vec{F}$.

$$W=\vec{F}\cdot\frac{\vec{r}(4)}{\|\vec{r}\|}=\frac{-2\vec{q}\cdot\vec{r}(4)}{\|\vec{r}\|}=\frac{-8(-7)}{\sqrt{6}}=\frac{56}{\sqrt{6}} \text{ Joules}$$

4. [12] Set $\vec{u}=\vec{i}-2\vec{j}+3\vec{k}$, $\vec{v}=2\vec{i}-\vec{j}+\vec{k}$ and $\vec{w}=2\vec{i}-\vec{j}-\vec{k}$. a) Find the area of the parallelogram having $\vec{v}$ and $\vec{w}$ as adjacent sides. b) Find the volume of the parallelepiped having $\vec{u}$, $\vec{v}$ and $\vec{w}$ as adjacent edges.

a) $A = \|\vec{v}\times\vec{w}\|$. $\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 2 & -1 & -1 \end{vmatrix} = (1+1)\vec{i} - (-2-2)\vec{j} + (-2+2)\vec{k} = 2\vec{i} + 4\vec{j}$

Hence

$A = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}\ \text{unit}^2$

b) $V = |\vec{u}\cdot(\vec{v}\times\vec{w})| = |1(2) - 2(4)| = |-6| = 6\ \text{unit}^3$

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5. [20] a) Show that the two lines $L_1 : x = 1-3t,\quad y = 4+2t,\quad z = 4+3t$, and $L_2 : x = 3+t,\quad y = 4-2t,\quad z = 3-2t$ intersect, and find their point of intersection A.

b) Find an equation for the plane $\mathcal{P}$ that contains both L_1 and L_2.

c) Find the distance between the plane $\mathcal{P}$ and the point $C(1,-2,-3)$.

a) Do we have t_1 and t_2 with

$1-3t_1 = 3+t_2$ (1)

$4+2t_1 = 4-2t_2$ (2)

$4+3t_1 = 3-2t_2$? (3)

multiply (1) by 2 and add result to (2) to get

$2-6t_1+4+2t_1 = 6+4 \to -4t_1 = 4 \to t_1 = -1$ (4)

use (4) in (1) to get; $4 = 3+t_2 \to t_2 = 1$ (5)

(4) & (5) in (3) yield; LHS $= 4-3 = 1 = 3-2 =$ RHS; so L_1 & L_2 intersect at $A(4,2,1)$.

b) $\vec{u}_1 = \langle -3, 2, 3\rangle \parallel L_1$, $\vec{u}_2 = \langle 1, -2, -2\rangle \parallel L_2$. So

$\vec{n} = \vec{u}_1 \times \vec{u}_2 =$ a normal to $\mathcal{P}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & 3 \\ 1 & -2 & -2 \end{vmatrix} = (-4+6)\vec{i} - (6-3)\vec{j} + (6-2)\vec{k} = 2\vec{i} - 3\vec{j} + 4\vec{k}$; A lies on $\mathcal{P}$

Equation of plane $2(x-4) - 3(y-2) + 4(z-1) = 0$ or

$2x - 3y + 4z - 6 = 0$

c) $d(C,\mathcal{P}) = \dfrac{|2(1) - 3(-2) + 4(-3) - 6|}{\sqrt{4+9+16}} = \dfrac{10}{\sqrt{29}}$

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6. [4] Find an equation and identify the surface that results when the cone $z = \sqrt{3x^2+3y^2}$ is reflected about the plane:

i) $z = 0$, ii) $x = z$.

i) Change z to $-z$; $-z = \sqrt{3x^2+3y^2}$ or $z = -\sqrt{3x^2+3y^2}$; upside down cone along z-axis

ii) switch x and z: $x = \sqrt{3z^2+3y^2}$; cone along x-axis

7 [6]. a) Convert from rectangular to spherical coordinates: i) $(3, -\sqrt{3}, -2)$. ii) Convert the equation $\theta = \frac{\pi}{4}$ from cylindrical to rectangular coordinates, and identify the surface.

i) $\rho = \sqrt{9+3+4} = \sqrt{16} = 4$

$\left.\begin{array}{l} \tan\theta = -\frac{\sqrt{3}}{3} \\ \theta \text{ in QIV} \end{array}\right\} \theta = \frac{11\pi}{6}$

$\cos\phi = -\frac{2}{4} = -\frac{1}{2} \to \phi = \frac{2\pi}{3}$

$\left(4, \frac{11\pi}{6}, \frac{2\pi}{3}\right)$

ii) $\tan\theta = 1 = \frac{y}{x}$

or $y = x$ or $y - x = 0$; plane

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8. [9] a) Find the points of intersection of the line $L: x = 1+t, \quad y = 2-t, \quad z = 5$ and the paraboloid $z = x^2 + y^2$.
b) Find an equation for the plane that contains both the line L from part a) and the point $D(2,3,4)$.

a) $5 = (1+t)^2 + (2-t)^2 = 1 + 2t + t^2 + 4 - 4t + t^2 = 5 - 2t + 2t^2$ or

$2t(-1+t) = 0$; so $t = 0$ or $t = 1$; so points of intersection are:

for $t = 0$: $(1, 2, 5)$

for $t = 1$: $(2, 1, 5)$

b) The point $B(1, 2, 5)$ lies on L, so B lies on the plane.

$\vec{u} = \langle 1, -1, 0 \rangle \parallel L$

$\vec{v} = \vec{BD} = \langle 2-1, 3-2, 4-5 \rangle = \langle 1, 1, -1 \rangle \parallel$ plane

$\vec{n} = \vec{u} \times \vec{v} =$ a normal to plane

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (1-0)\vec{i} - (-1-0)\vec{j} + (1+1)\vec{k} = \vec{i} + \vec{j} + 2\vec{k}$$

Equation for plane: $x - 1 + y - 2 + 2(z-5) = 0$.