Orbital Mechanics

These notes provide an alternative and elegant derivation of Kepler's three laws for the motion of two bodies resulting from their gravitational force on each other.

Orbit Equation and Kepler I

Consider the equation of motion of one of the particles (say, the one with mass m) with respect to the other (with mass M), i.e. the relative motion of m with respect to M:

$$\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3}, \tag{1}$$

with μ given by

$$\mu = G(M+m). \tag{2}$$

Let $\vec{h}$ be the *specific* angular momentum (i.e. the angular momentum per unit mass) of m,

$$\vec{h} = \vec{r} \times \dot{\vec{r}}. \tag{3}$$

The $\times$ sign indicates the cross product. Taking the derivative of $\vec{\mathrm{h}}$ with respect to time, t, we can write

$$\begin{aligned} \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) &= \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned} \tag{4}$$

The first term of the right hand side is zero for obvious reasons; the second term is zero because of Eqn. 1: the vectors $\vec{r}$ and $\ddot{\vec{r}}$ are antiparallel. We conclude that $\vec{h}$ is a constant vector, and its magnitude, h, is constant as well. The vector $\vec{h}$ is perpendicular to both $\vec{r}$ and the velocity $\dot{\vec{r}}$, hence to the plane defined by these two vectors. This plane is the orbital plane.

Let us now carry out the cross product of $\ddot{\vec{r}}$, given by Eqn. 1, and $\vec{h}$, and make use of the vector algebra identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \tag{5}$$

to write

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^3}\left((\vec{r} \cdot \dot{\vec{r}})\vec{r} - r^2\dot{\vec{r}}\right). \tag{6}$$

The $\vec{r} \cdot \dot{\vec{r}}$ in this equation can be replaced by $r\dot{r}$ since

$$\vec{r} \cdot \vec{r} = r^2,$$

and after taking the time derivative of both sides,

$$\begin{aligned} \frac{d}{dt}(\vec{r} \cdot \vec{r}) &= \frac{d}{dt}(r^2), \\ 2\vec{r} \cdot \dot{\vec{r}} &= 2r\dot{r}, \\ \vec{r} \cdot \dot{\vec{r}} &= r\dot{r}. \end{aligned} \tag{7}$$

Substituting Eqn. 7 into Eqn. 6 gives

$$\begin{aligned} \ddot{\vec{r}} \times \vec{h} &= -\frac{\mu}{r^3}\left(r\dot{r}\vec{r} - r^2\dot{\vec{r}}\right), \\ &= \mu\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}\right), \\ &= \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right). \end{aligned} \tag{8}$$

Integrating the latter equation and considering $\vec{h}$ is constant we get

$$\dot{\vec{r}} \times \vec{h} = \mu\left(\frac{\vec{r}}{r} + \vec{e}\right), \tag{9}$$

where the vector $\vec{e}$ is an integration constant called the Laplace-Runge-Lenz (LRL) vector. It is clear that the vectors $\vec{e}$ and $\vec{h}$ are perpendicular to one another. Hence, $\vec{e}$ must be a vector in the orbital plane!

Finally, let us make the scalar product of both sides of Eqn. 9 with the vector $\vec{r}$. We have

$$\begin{aligned} \vec{r} \cdot (\dot{\vec{r}} \times \vec{h}) &= \mu\left(\frac{\vec{r} \cdot \vec{r}}{r} + \vec{e} \cdot \vec{r}\right) \\ &= \mu\left(r + re\cos\theta\right), \end{aligned} \tag{10}$$

where θ is the angle between the vectors $\vec{r}$ and $\vec{e}$. Applying the vector algebra identity

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \tag{11}$$

to the left-hand side of Eqn. 10 we get

$$\begin{aligned} \vec{r} \cdot (\dot{\vec{r}} \times \vec{h}) &= \vec{h} \cdot (\vec{r} \times \dot{\vec{r}}), \\ &= \vec{h} \cdot \vec{h}, \\ &= h^2. \end{aligned} \tag{12}$$

Replacing the left hand side of Eqn. 10 by Eqn. 12 then gives

$$h^2 = \mu\,(r + re\cos\theta)\,,$$

or

$$\boxed{r = \frac{h^2/\mu}{1 + e\cos\theta}}\,. \tag{13}$$

In analytical geometry, the general equation of an ellipse in polar coordinates, r and θ, with one of the ellipse's foci as the origin of the coordinate frame (see Figure 3.6 and Eqution 3.42 in the Ryden-Peterson textbook), is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}, \tag{14}$$

The distance r is the magnitude of the position vector $\vec{r}$, which makes an angle θ with the reference axis along the line of apsides. This angle is called the *true anomaly*. The quantity a is called the *semimajor axis* and is half the length of the largest diameter of the ellipse, called the *major axis*. The two foci are located on the major axis and are equidistant from the center of the ellipse. That distance is equal to ae. For $e = 0$, $r = a$, and the curve is a circle with radius a. The two foci of a circle coincide with the center of the circle. Note that the periapse distance, r_p, and apoapse distance, r_a, are obtained by entering $f = 0$ and $f = \pi$, respectively, in Eqn. 14. Doing so we get

$$r_p = a(1 - e), \tag{15}$$

and

$$r_a = a(1 + e). \tag{16}$$

Comparing Eqns. 13 and 14, we conclude that the orbit of m around M is a conic section, with a semi major axis a and eccentricity e related to h and μ via the equation

$$\frac{h^2}{\mu} = a(1 - e^2),$$

or

$$\boxed{h = \sqrt{\mu a(1 - e^2)}}\,. \tag{17}$$

The magnitude e of the LRL vector $\vec{e}$ is the *eccentricity* of the conic section. For $0 \leq e < 1$, the conic section is an *ellipse*. In that case, the curve is closed and the mass m describes a closed orbit around the attracting mass M, located at one of the foci of the ellipse. What value of the angle θ makes r a minimum? The answer is, of course, that value of θ that makes $1 + e\cos\theta$ a maximum, which is when $\cos\theta = +1$, or $\theta = 0$, i.e. when $\vec{r}$ is parallel to $\vec{e}$. Thus, the LRL vector $\vec{e}$ is a vector that points from the point of central attraction to the point of closest approach, the *periapse* point. The opposite point on the ellipse, when $\theta = \pi$, is called the *apoapse* point.[1]

For $e = 1$, $r \to \infty$ as $\theta \longrightarrow \pm\pi$, which describes a *parabola*. For $e > 1$, the orbit is a *hyperbola*. In this case $r \longrightarrow \infty$ along asymptotes defined by values of $\theta = \theta_\infty < \pi$ and given by $e \cos\theta_\infty = -1$. For $e \geq 1$, Equation 13 holds unchanged, and parabolic or hyperbolic orbits do occur in nature. For example, non-periodic comets describe hyperbolic orbits around the Sun; they approach the Sun, swing by once, and then move away along their hyperbolic path, to never come back. For parabolas and hyperbolas, however, the geometric description of Eqn. 14 takes on a slightly different form. From here on,we restrict ourselves to elliptical orbits unless specifically stated otherwise.

Kepler II

Let us now consider a right-handed, Cartesian coordinate frame with origin O at the center of mass of the M, m system, and with the x, y-plane coinciding with the orbital plane. We also consider a system of polar coordinates $(r,\, \theta)$, with origin at M, and a system of two orthogonal, corotating unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ with cartesian coordinates $(\cos\theta, \sin\theta, 0)$ and $(-\sin\theta, \cos\theta, 0)$, respectively. I refer to Section 3.1.1 in Ryden & Peterson for specifics and figures. The velocity $\vec{v} = \dot{\vec{r}}$ can be written as

$$\vec{v} = v_{\mathrm{R}}\hat{\mathbf{r}} + v_{\mathrm{T}}\hat{\boldsymbol{\theta}}, \tag{18}$$

with v_{R} and v_{T} the radial and tangential components of $\vec{v}$, respectively. It is shown on page 64 of Ryden & Peterson that

$$v_{\mathrm{R}} = \dot{r}, \tag{19}$$

and

$$v_{\mathrm{T}} = r\dot{\theta}. \tag{20}$$

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[1] Both these labels can be modified to include the name of the attracting body. For example, for motion around the Sun, we refer to the point of closest approach as the *perihelion* point. For the Moon and other satellites of the Earth we call this point the *perigee*. In a binary star system, the point of closest approach of one star as it orbits the other is called the *periastron* point. Similarly, the point of maximum distance would be called the *aphelion*, *apogee*, *apastron*.

In terms of the corotating $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{k}}$ frame, the specific angular momentum vector $\vec{h}$ can be written as

$$\begin{aligned}
\vec{h} &= \vec{r} \times \dot{\vec{r}} \\
&= \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{k}} \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}} + r^2\dot{\theta}\hat{\mathbf{k}} \\
&= r^2\dot{\theta}\hat{\mathbf{k}}. \qquad (21)
\end{aligned}$$

Since $\vec{h}$ is a constant vector, $r^2\dot{\theta}$ is constant. Consider the position vector $\vec{r}$ sweeping an area dA as the mass m moves in its orbit from the position at time t to the position at time $t+dt$. dA can be considered to be the area of an infinitesimal triangle with sides r and $rd\theta$, so we can write

$$dA = \frac{1}{2}r^2 d\theta, \qquad (22)$$

or

$$\begin{aligned}
dA &= \frac{1}{2}r^2\dot{\theta}dt \\
&= \frac{1}{2}hdt. \qquad (23)
\end{aligned}$$

Integrating this from time t_1 to time t_2, when m is at position 1 and 2, respectively, the area A swept by the position vector is

$$A = \frac{1}{2}h(t_2 - t_1). \qquad (24)$$

Hence, equal Δt's give equal A's, which is Kepler's second law, the law of areas: the position vector sweeps out equal areas in equal intervals of time.

Kepler III

In Eqn. 24, let $\Delta t = t_2 - t_1$ be the time for one complete revolution of m around M. This time interval is called the *period* of the orbital motion. Let P be this period. The corresponding area A swept by the position vector must then be the area of the entire ellipse, given by the equation

$$A = \pi ab, \qquad (25)$$

with a the semimajor axis and b the semiminor axis of the ellipse. The latter is related to the former via the eccentricity e:

$$b = a\sqrt{1 - e^2}. \qquad (26)$$

For $\Delta t = P$, Eqn. 24 then becomes

$$\pi a^2 \sqrt{1-e^2} = \frac{1}{2} h P, \tag{27}$$

and with Eqn. 17,

$$\pi a^2 \sqrt{1-e^2} = \frac{1}{2} \sqrt{\mu a(1-e^2)}\, P. \tag{28}$$

The $\sqrt{1-e^2}$ drops out, and after squaring both sides and rearranging variables we get Newton's form of Kepler's third law:

$$\boxed{\mu = G(M+m) = 4\pi^2 \frac{a^3}{P^2}}\,. \tag{29}$$

The Vis Viva Equation

The following is an alternative derivation of Leibniz' vis viva equation, the important Equation 3.67 in Ryden & Peterson.

The magnitude v of the velocity $\vec{v}$ of m with respect to M can be written as

$$v^2 = {v_\mathrm{R}}^2 + {v_\mathrm{T}}^2, \tag{30}$$

or, using Eqns. 19 and 20, as

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2. \tag{31}$$

In here, the $\dot{r}$ can be obtained from differentiating Eqn. 14, which leads to

$$\begin{aligned} \dot{r} &= a(1-e^2) \sin\theta\, \dot{\theta}\, (1 + e\cos\theta)^{-2} \\ &= \frac{r^2}{a(1-e^2)}\, e \sin\theta\, \frac{\sqrt{\mu a(1-e^2)}}{r^2} \\ &= \sqrt{\frac{\mu}{a(1-e^2)}}\, e \sin\theta. \end{aligned} \tag{32}$$

The $\dot{\theta}$ comes from Eqn. 21:

$$\begin{aligned} \dot{\theta} &= \frac{h}{r^2} \\ &= \frac{\sqrt{\mu a(1-e^2)}}{r^2} \end{aligned} \tag{33}$$

The vis viva equation then follows by substituting Eqns. 32 and 33 into 31 and carrying out some algebra:

$$\boxed{v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)}\,. \tag{34}$$

This is a very important equation. It tells us that, for given masses M and m, the orbital speed only depends on the distance r between the two bodies and the orbit's semi major axis.

Applying vis viva at the periapse point, with r given by Eqn. 15, yields the orbital speed at periapse passage,

$$v_p = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}}, \tag{35}$$

which corresponds to the maximum value v can have. Similarly, vis viva and Eqn. 16 give the orbital speed at apoapse,

$$v_a = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1-e}{1+e}}, \tag{36}$$

which is the minimum orbital speed of m. Note that the product

$$v_p v_a = \frac{\mu}{a} \tag{37}$$

is independent of the eccentricity e.

Energy

The total mechanical energy of the system of two bodies (M, m) is the sum of the kinetic energy of M, the kinetic energy of m, and the gravitational potential energy of the (M, m) system. Choose a coordinate frame with origin at the center of mass (CM) of the system. Vectors $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of M and m, respectively. Clearly, the relative position of m with respect to M is

$$\vec{r} = \vec{r}_2 - \vec{r}_1. \tag{38}$$

Because the position vector of the CM is the zero vector (CM is at O), and using the definition of the CM, we have

$$M\vec{r}_1 + m\vec{r}_2 = \vec{0}, \tag{39}$$

or

$$M\vec{r}_1 = -m\vec{r}_2. \tag{40}$$

Combining the latter with 38 gives

$$\vec{r}_1 = -\frac{m}{M+m}\vec{r}, \tag{41}$$

and after taking the time derivative,

$$\vec{v}_1 = -\frac{m}{M+m}\vec{v}. \tag{42}$$

Likewise we obtain

$$\vec{v}_2 = +\frac{M}{M+m}\vec{v}. \tag{43}$$

The total mechanical energy E then becomes:

$$\begin{aligned} E &= \frac{1}{2}M{v_1}^2 + \frac{1}{2}m{v_2}^2 - G\frac{Mm}{r} \\ &= \frac{1}{2}M\left(\frac{m}{M+m}\right)^2 v^2 + \frac{1}{2}m\left(\frac{M}{M+m}\right)^2 v^2 - G\frac{Mm}{r} \\ &= \frac{1}{2}\left(\frac{Mm}{M+m}\right)v^2 - G\frac{Mm}{r}. \end{aligned} \tag{44}$$

The quantity in parentheses, $Mm/(M+m)$, has the dimension of mass and is called the *reduced mass* of the system. Substituting v^2 by the vis viva Eqn. 34, and after some algebra, we get

$$E = -\frac{1}{2}G\frac{Mm}{a}, \tag{45}$$

or, given $\mu = G(M+m)$,

$$E = -\frac{1}{2}\left(\frac{Mm}{M+m}\right)\frac{\mu}{a}. \tag{46}$$

Note that we have used the viv viva equation for *eliptical* motion, so Eqn. 45 gives the total energy of the system for elliptical orbits. Given the masses M and m, E is uniquely determined by the semimajor axis a of the orbit. Also, E is negative, indicating a bound system.

For parabolic orbits, both the vis viva (Eqn. 34) and energy (Eqn. 45) equations are still valid, provided we set $a = \infty$. We then have

$$E_{\rm parab} = 0, \tag{47}$$

and the vis viva equation becomes

$$v^2_{\rm para} = \frac{2G(M+m)}{r}. \tag{48}$$

A particle m moving in a parabolic orbit with this parabolic speed $v = \sqrt{2G(M+m)/r}$ will make it to infinity, i.e. will "escape" the gravitational pull of M. This speed is referred to as the *escape speed*, $v_{\rm esc}$. If $m << M$, the escape speed reduces to

$$v_{\rm esc} = \sqrt{\frac{2GM}{r}}, \tag{49}$$

which is Eqn. 3.62 in Ryden & Peterson.

For hyperbolic orbits, the total energy is positive and given by

$$E = \frac{1}{2}G\frac{Mm}{a}. \tag{50}$$

Virial Theorem

The $-GMm/a$ part of Eqn. 45 represents the "mean potential energy" of the system, with the mean taken over one orbital cycle. Thus for a two-body orbit we find the total energy to be equal to half the time-averaged potential energy. This is the so-called *virial theorem* for a gravitationally bound system of many particles. The theorem can be expressed as

$$< E >= \frac{1}{2} < U >, \tag{51}$$

or, since

$$< E >=< K > + < U >, \tag{52}$$

as

$$< K >= -\frac{1}{2} < U > . \tag{53}$$

A rigorous derivation of the virial theorem is in Ryden & Peterson Section 3.4.

The virial theorem is very useful in astronomy in the study of large stellar system such as star clusters and galaxies, It also plays an important role during the star formation process. When part of a nebula (a cloud of interstellar gas and dust) collapses gravitationally, its (negative) potential energy decreases (the inter-particle distance decreases, hence the absolute value of the PE increases, but since PE is a negative number, the PE decreases). According to the virial theorem, half of the lost PE goes into KE of the particles, i.e. the internal energy (and temperature) of the collapsing blob increases. What happens to the other half? That other half is carried out of the blob of collapsing gas by photons, i.e. it gets radiated away by light. So, during star formation, *before* the onset of nuclear fusion, stars in the process of forming already shine.

Interplanetary Travel

We now have all the ingredients for launching probes and get them from one orbit to another around the same planetary body, or from one planet to another: see the idea of Hohmann transfer orbits in Ryden & Peterson, Section 3.3, and our discussion in class.