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On the asymptotic behavior of solutions to a structural acoustics model

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Abstract

This article is concerned with the long term behavior of solutions to a structural acoustics model consisting of a semilinear wave equation defined on a bounded domain $\Omega \subset \mathbb{R}^3$ which is coupled with a Berger plate equation acting on a flat portion of the boundary of Ω. The system is influenced by several competing forces, in particular a source term acting on the wave equation which is allowed to have a supercritical exponent.

Our results build upon those obtained by Becklin and Rammaha [8]. With some restrictions on the parameters in the system and with careful analysis involving the Nehari manifold we obtain global existence of potential well solutions and establish either exponential or algebraic decay rates of energy, dependent upon the behavior of the damping terms. The main novelty in this work lies in our stabilization estimate, which notably does not generate lower-order terms. Consequently, the proof of the main result is shorter and more concise.

MSC: 35L70

Keywords: Structural acoustics models; Wave-plate models; Potential well solutions; Uniform decay rates

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1. Introduction

1.1. The model

Structural acoustic interaction models have rich history. These models are well known in both the physical and mathematical literature and go back to the canonical models considered in [6,24]. For instance, the model studied by Beale [6] is constructed as follows. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain filled with a fluid which is at rest, except for an acoustic wave motion. If $u(x,t)$ is the velocity potential of the fluid, so that $-\nabla u(x,t)$ is the particle velocity, then u satisfies the wave equation

$$u_{tt} = c^2 \Delta u \quad \text{in } \Omega \times (0,T),$$

where c is the speed of sound in the medium. Further assume that $\Gamma = \partial\Omega$ is not rigid but subject to small oscillations, where each point $x \in \partial\Omega$ reacts to the pressure wave like a damped oscillator. Then, the normal displacement w of the boundary into the domain satisfies the ODE:

$$m(x)w_{tt}(x,t) + d(x)w_t(x,t) + k(x)w(x,t) = -\rho u_t(x,t)|_\Gamma \quad \text{on } \Gamma \times (0,T),$$

where ρ is the density of the fluid. In addition, by assuming the boundary Γ is impenetrable, then from the continuity of velocity at the boundary one has:

$$\partial_\nu u = w_t \quad \text{on } \Gamma \times (0,T).$$

Motivated by the above model, we study a structural acoustics model influenced with nonlinear forces. Precisely, we study the coupled system of PDEs:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + g_2(w_t) + u_t|_\Gamma = h(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), & \end{cases} \tag{1.1}$$

where the initial data reside in the finite energy space, i.e.,

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \text{ and } (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma),$$

where the space $H^1_{\Gamma_0}(\Omega)$ is defined in (2.1) below.

Here, $\Omega \subset \mathbb{R}^3$ is a bounded, open, connected domain with a smooth boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma}$, where Γ_0 and Γ are two disjoint, open, connected sets of positive Lebesgue measure. Moreover, Γ is a *flat* portion of the boundary of Ω and is referred to as the elastic wall. The part Γ_0 of the boundary $\partial\Omega$ describes a rigid wall, while the coupling takes place on the flexible wall Γ.

It is interesting that problem (1.1) includes geometric elements of one, two and three dimensions. In particular, Ω is a three-dimensional region in which a nonlinear wave equation for u is defined. The boundary of Ω contains a flat portion Γ, a two-dimensional plane, in which a plate equation for w is defined. Finally, the boundary of Γ is a smooth curve $\partial\Gamma$, which is a one-dimensional geometric object, on which a boundary condition for w is imposed.

The nonlinearities f and h represent source terms acting on the wave and plate equations respectively, where $f(u)$ is of a supercritical order and both source terms are allowed to have "bad" signs which may cause instability (blow up) in finite time. In addition, the system is influenced by two other competing forces, namely $g_1(u_t)$ and $g_2(w_t)$ representing frictional damping terms acting on the wave and plate equations, respectively. The presence of frictional damping is necessary to stabilize the system; otherwise, nonlinear source terms can lead to blow up in finite time. The vectors ν and ν_Γ denote the outer normals to Γ and $\partial\Gamma$; respectively.

Models such as (1.1) arise in the context of modeling gas pressure in an acoustic chamber which is surrounded by a combination of rigid and flexible walls. The pressure in the chamber is described by the solution to a wave equation, while vibrations of the flexible wall are described by the solution to a coupled a Berger plate equation. We refer the reader to [13] and the references quoted therein on the Berger model.

In system (1.1), the coupling of the wave equation for u and the plate equation for w are through the term $u_t|_\Gamma$. Note, the solution (u, u_t) for the wave equation belongs to the finite energy space $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$. Therefore, u_t belongs to the space $L^2(\Omega)$, and in general, without additional regularity, one cannot take the trace of an $L^2(\Omega)$ function on Γ. Therefore, in the definition of weak solutions below, namely, Definition 2.4, we test $u_t|_\Gamma$ with a test function ψ, and pass the time derivative to ψ in the spirit of weak derivative, and obtain the term $\int_\Gamma u|_\Gamma(t)\psi_t(t)d\Gamma - \int_\Gamma u|_\Gamma(0)\psi_t(0)d\Gamma - \int_0^t \int_\Gamma u|_\Gamma \psi_t d\Gamma d\tau$, which makes perfect sense since $u \in H^1_{\Gamma_0}(\Omega)$ regular enough to take the boundary trace on Γ. Moreover, in the paper [7], Becklin and Rammaha studied a similar model with restoring source terms but no damping term, and they pointed out that there is a hidden regularity for u_t on Γ, which was $u_t|_{\Gamma\times(0,T)} \in H^{-2/3}(\Gamma \times (0, T))$ by assuming additional regularity on initial data and that the source term was subcritical. The point is that the term $u_t|_\Gamma$ is an important element connecting the dynamics of u and w and can cause difficulties in the analysis of model (1.1).

One of the novelties of this manuscript lies in that the important stabilization estimate (4.14) presented in Lemma 4.1 does not contain lower-order terms on the right-hand side of the inequality. This feature greatly shortens the proof, because the standard lengthy compactness-uniqueness argument to absorb the lower-order terms is completely avoided. In particular, the estimate in subsection 4.14 includes some new idea and is applicable to the study of energy decay of other related systems. Such type of refined estimate was originally introduced in paper [22] by Guo, Rammaha and Sakuntasathien for a similar purpose of avoiding lower-order terms.

1.2. Literature overview

As we mentioned earlier, structural acoustic interaction models have rich history and they go back to the canonical models considered in [6,24]. In the context of stabilization and controllability of structural acoustic models there is a very large body of literature. We refer the reader to the monograph by Lasiecka [28] which provides a comprehensive overview and quotes many works on these topics. Other related contributions worthy of mention include [1–4,12,16,17,27].

This manuscript focuses on potential well solutions of system (1.1). The study of potential well solutions for nonlinear hyperbolic equations has a long history. For example, Payne and Sattinger [32] considered a nonlinear hyperbolic equation in the canonical form:

$$u_{tt} = \Delta u + f(u), \text{ with } u(0) = u_0, \ u_t(0) = u_1, \tag{1.2}$$

such that $u = 0$ on the boundary of a smooth bounded domain $\Omega \subset \mathbb{R}^n$. An important quantity for equation (1.2) is the potential energy functional $J(u) = \frac{1}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega F(u)dx$ where $F(u) = \int_0^u f(s)ds$. The depth of the potential well d can be defined as the mountain pass level, i.e., $d = \inf_{u\neq 0}\sup_{\lambda>0} J(\lambda u)$. Assume the initial total energy and the initial potential energy $J(u_0)$ are both less than d. Then, if $\frac{d}{d\lambda}J(\lambda u) \geq 0$ for $0 < \lambda \leq 1$, i.e., J is increasing along any ray from the origin, then the weak solution is global in time; but, if $\|u\|^2_{L^2} < \int_\Omega uf(u)dx$, the solution blows up in finite time (see [32]). The global existence of potential well solutions for (1.2) can also be found in an earlier work by Sattinger [36]. In the same spirit, we show the global existence of potential well solutions for the structural acoustics model (1.1), and the blow-up of solutions will be considered in a future paper.

System (1.1) involves competing forces. In particular, nonlinear source terms $f(u)$ and $h(w)$ are competing with nonlinear frictional-type damping terms $g_1(u_t)$ and $g_2(w_t)$. A classical result on wave equations with nonlinear damping term $|u_t|^{m-1}u_t$ and source term $|u|^{p-1}u$ was established by Georgiev and Todorova [15] in 1994. In particular, if the damping term dominates the source term ($m \geq p$), then weak solutions are global in time; whereas, if the source term surpasses the damping term $p > m$, then weak solutions may blow up if initial energy is large enough. An extension to a supercritical source term ($3 \leq p < 6$ in 3D) was obtained by Bociu and Lasiecka [9–11]. Furthermore, for a nonlinear wave equation with a source term of rapid polynomial growth rate (including the range $p \geq 6$ in a 3D periodic domain), Guo showed in [18] the global well-posedness of weak solutions, provided the damping term has sufficiently fast growth rate ($m \geq \frac{3}{2}p - \frac{5}{2}$ if $p \geq 6$). There are many other interesting works concerned with the competition of source terms and various types of damping terms in nonlinear wave equations. See, for example, [5,20,21,23,25,26,33,34] and references therein.

Recently, Vicente studied a wave equation with viscoelastic acoustic boundary condition and a boundary source term in paper [38]. The model studied in [38] has some similarity with our system (1.1). But there are some major differences. In particular, on the boundary, the dynamics of our system (1.1) is described by a Berger plate equation, in contrast to the viscoelastic acoustic boundary condition in [38]. Berger plate equation models vibrations of the flexible wall of an acoustic chamber, whereas, viscoelastic acoustic boundary condition describes vibrations with delay. Another difference is that our system (1.1) includes an “interior” source term $f(u)$. By assuming the interior source term $f(u)$ has sufficiently high growth rate, we investigate the blow-up phenomenon of system (1.1) in our recent paper [14].

2. Preliminaries and main results

2.1. Notation

Throughout the paper the following notational conventions for L^p space norms and standard inner products will be used:

$$||u||_p = ||u||_{L^p(\Omega)}, \qquad (u, v)_\Omega = (u, v)_{L^2(\Omega)},$$
$$|u|_p = ||u||_{L^p(\Gamma)}, \qquad (u, v)_\Gamma = (u, v)_{L^2(\Gamma)}.$$

We also use the notation γu to denote the *trace* of u on Γ.

Further, we put

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}. \tag{2.1}$$

It is well-known that the standard norm $\|u\|_{H^1_{\Gamma_0}(\Omega)}$ is equivalent to $\|\nabla u\|_2$. Thus, we put:

$$\|u\|_{H^1_{\Gamma_0}(\Omega)} = \|\nabla u\|_2\,,\ (u, v)_{H^1_{\Gamma_0}(\Omega)} = (\nabla u, \nabla v)_\Omega. \tag{2.2}$$

For a similar reason, we put:

$$\|w\|_{H^2_0(\Gamma)} = |\Delta w|_2\,,\ (w, z)_{H^2_0(\Gamma)} = (\Delta w, \Delta z)_\Gamma. \tag{2.3}$$

Let Y be a Banach space. We denote the duality pairing between the dual space Y' and Y by $\langle \cdot, \cdot \rangle$.

Throughout the paper, the following Sobolev imbeddings will be used: $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$, $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^4(\Gamma)$, and $H^1(\Gamma) \hookrightarrow L^q(\Gamma)$ for all $1 \le q < \infty$.

2.2. *Weak solutions*

Throughout the paper, we study (1.1) under the following assumptions.

Assumption 2.1.

Damping terms $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ *are continuous and monotone increasing functions with* $g_1(0) = g_2(0) = 0$. *In addition, the following growth conditions at infinity hold: there exist positive constants* α *and* β *such that, for* $|s| \ge 1$,

$$\alpha |s|^{m+1} \le g_1(s)s \le \beta |s|^{m+1},\ \textit{with } m \ge 1,$$
$$\alpha |s|^{r+1} \le g_2(s)s \le \beta |s|^{r+1},\ \textit{with } r \ge 1.$$

Source terms f *and* h *are functions in* $C^1(\mathbb{R})$ *such that*

$$|f'(s)| \le C(|s|^{p-1} + 1),\ \textit{with } 1 \le p < 6,$$
$$|h'(s)| \le C(|s|^{q-1} + 1),\ \textit{with } 1 \le q < \infty.$$

Parameters $p\frac{m+1}{m} < 6$.

Remark 2.2. As the following bounds will be used often throughout the paper, it is worthy of note that the above assumption implies that

$$\begin{cases} |f(u)| \leq C(|u|^p + 1), \quad |f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)|u - v|, \\ |h(w)| \leq C(|w|^q + 1), \; |h(w) - h(z)| \leq C(|w|^{q-1} + |z|^{q-1} + 1)|w - z|. \end{cases} \tag{2.4}$$

The following assumption will be needed for establishing the result on the uniqueness of solutions.

Assumption 2.3. *For $p > 3$, we assume that $f \in C^2(\mathbb{R})$ with $|f''(u)| \leq C(|u|^{p-2} + 1)$ for all $u \in \mathbb{R}$.*

We begin by introducing the definition of a suitable weak solution for (1.1).

Definition 2.4. A pair of functions (u, w) is said to be a *weak solution* of (1.1) on the interval $[0, T]$ provided:

(i) $u \in C([0, T]; H^1_{\Gamma_0}(\Omega))$, $u_t \in C([0, T]; L^2(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$,
(ii) $w \in C([0, T]; H^2_0(\Gamma))$, $w_t \in C([0, T]; L^2(\Gamma)) \cap L^{r+1}(\Gamma \times (0, T))$,
(iii) $(u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$,
(iv) $(w(0), w_t(0)) = (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma)$,
(v) The functions u and w satisfy the following variational identities for all $t \in [0, T]$:

$$\begin{aligned} &(u_t(t), \phi(t))_\Omega - (u_1, \phi(0))_\Omega - \int_0^t (u_t(\tau), \phi_t(\tau))_\Omega d\tau + \int_0^t (\nabla u(\tau), \nabla\phi(\tau))_\Omega d\tau \\ &\quad - \int_0^t (w_t(\tau), \gamma\phi(\tau))_\Gamma d\tau + \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau) dx d\tau \\ &\quad = \int_0^t \int_\Omega f(u(\tau))\phi(\tau) dx d\tau, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &(w_t(t) + \gamma u(t), \psi(t))_\Gamma - (w_1 + \gamma u(0), \psi(0))_\Gamma - \int_0^t (w_t(\tau), \psi_t(\tau))_\Gamma d\tau \\ &\quad - \int_0^t (\gamma u(\tau), \psi_t(\tau))_\Gamma d\tau + \int_0^t (\Delta w(\tau), \Delta\psi(\tau))_\Gamma d\tau \\ &\quad + \int_0^t \int_\Gamma g_2(w_t(\tau))\psi(\tau) d\Gamma d\tau = \int_0^t \int_\Gamma h(w(\tau))\psi(\tau) d\Gamma d\tau, \end{aligned} \tag{2.6}$$

for all test functions ϕ and ψ satisfying: $\phi \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap L^{m+1}(\Omega \times (0, T))$, $\psi \in C\left([0, T]; H^2_0(\Gamma)\right)$ with $\phi_t \in L^1(0, T; L^2(\Omega))$, and $\psi_t \in L^1(0, T; L^2(\Gamma))$.

As mentioned earlier, our work in this paper is based on the existence results which were established in [8]. For the reader's convenience, we first summarize the important results in [8].

Theorem 2.5 *(Local and global weak solutions [8]). Under the validity of Assumption 2.1, then there exists a local weak solution* (u, w) *to* (1.1) *in the sense of Definition 2.4, defined on* $[0, T_0]$ *for some* $T_0 > 0$ *depending on the initial quadratic energy* $E(0)$, *where the quadratic energy is defined as*

$$E(t) := \frac{1}{2}\left(\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2\right). \tag{2.7}$$

- (u, w) *satisfies the following energy identity for all* $t \in [0, T_0]$:

$$\begin{aligned} E(t) &+ \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau \\ &= E(0) + \int_0^t \int_\Omega f(u)u_t dx d\tau + \int_0^t \int_\Gamma h(w)w_t d\Gamma d\tau. \end{aligned} \tag{2.8}$$

- *In addition to Assumption 2.1, if we assume that* $u_0 \in L^{p+1}(\Omega)$, $p \le m$ *and* $q \le r$, *then the said solution* (u, w) *is a global weak solution and* T_0 *can be taken arbitrarily large.*
- *If Assumption 2.1 is valid, and if we additionally assume that* $u_0 \in L^{3(p-1)}(\Omega)$ *and* $m \ge 3p - 4$ *when* $p > 3$, *then weak solutions of* (1.1) *are unique.*
- *If Assumptions 2.1 and 2.3 are valid, and if we further assume that* $u_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$ *then, weak solutions of* (1.1) *are unique.*

2.3. Potential well

In this section we introduce the concepts of Nehari manifold and potential well.
Let us first impose additional assumptions on source terms f and h.

Assumption 2.6.

- *There exists a nonnegative function* $F(s) \in C^1(\mathbb{R})$ *such that* $F'(s) = f(s)$, *and* F *is homogeneous of order* $p + 1$, *i.e.,* $F(\lambda s) = \lambda^{p+1}F(s)$, *for* $\lambda > 0$, $u \in \mathbb{R}$.
- *There exists a nonnegative function* $H(s) \in C^1(\mathbb{R})$ *such that* $H'(s) = h(s)$, *and* H *is homogeneous of order* $q + 1$, *i.e.,* $H(\lambda s) = \lambda^{q+1}H(s)$, *for* $\lambda > 0$, $s \in \mathbb{R}$.

Remark 2.7. From Euler homogeneous function theorem we infer that

$$uf(u) = (p + 1)F(u), \quad wh(w) = (q + 1)H(w). \tag{2.9}$$

It follows from Assumption 2.1 that there exists a positive constant M such that

$$F(u) \le M(|u|^{p+1} + 1), H(w) \le M(|w|^{q+1} + 1), \tag{2.10}$$

which, noting that F and H are homogeneous, yields that

$$F(u) \leq M|u|^{p+1}, \quad H(w) \leq M|w|^{q+1}. \tag{2.11}$$

Moreover, we know that f is homogeneous of order p and h is homogeneous of order q, and we see from (2.9) and (2.11) that

$$|f(u)| \leq (p+1)M|u|^p, \quad |h(w)| \leq (q+1)M|w|^q. \tag{2.12}$$

Put $X = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$. According to (2.2) and (2.3), X is endowed by the natural norm:

$$\|(u, w)\|_X = (\|\nabla u\|_2^2 + |\Delta w|_2^2)^{1/2}. \tag{2.13}$$

Define the nonlinear functional $\mathcal{J} : X \to \mathbb{R}$ by

$$\mathcal{J}(u, w) := \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - \int_\Omega F(u)dx - \int_\Gamma H(w)d\Gamma. \tag{2.14}$$

Then, the *potential energy* of the system is given by $\mathcal{J}(u(t), w(t))$. The Fréchet derivative of $\mathcal{J}$ at $(u, w) \in X$ is given by

$$\langle \mathcal{J}'(u, w), (\phi, \psi)\rangle = \int_\Omega \nabla u \cdot \nabla\phi dx + \int_\Gamma \Delta w \cdot \Delta\psi d\Gamma - \int_\Omega f(u)\phi dx - \int_\Gamma h(w)\psi d\Gamma \tag{2.15}$$

for $(\phi, \psi) \in X$.

The *Nehari manifold* associated with the functional $\mathcal{J}$ is defined as

$$\mathcal{N} := \left\{(u, w) \in X\backslash\{(0, 0)\} : \langle \mathcal{J}'(u, w), (u, w)\rangle = 0\right\}. \tag{2.16}$$

The Nehari manifold is a manifold of functions, whose definition is motivated by the work of Zeev Nehari [30,31] for nonlinear second-order differential equations.

Due to (2.16), (2.15) and (2.9), we have

$$\mathcal{N} = \Big\{(u, w) \in X\backslash\{(0, 0)\} : \|\nabla u\|_2^2 + |\Delta w|_2^2 = (p+1)\int_\Omega F(u)dx + (q+1)\int_\Gamma H(w)d\Gamma\Big\}. \tag{2.17}$$

We define the *potential well* associated with the potential energy $\mathcal{J}(u, w)$ by

$$\mathcal{W} := \{(u, w) \in X : \mathcal{J}(u, w) < d\}, \tag{2.18}$$

where the *depth of the potential well* $\mathcal{W}$ is defined as

$$d := \inf_{(u,w)\in\mathcal{N}} \mathcal{J}(u, w). \tag{2.19}$$

In order to make sure that the set $\mathcal{W}$ is non-empty, we need to verify that d is strictly positive. The positivity of d is provided by the following lemma, under certain assumptions.

Lemma 2.8. *Let Assumption 2.1 and Assumption 2.6 hold. Let $1 < p \le 5$ and $q > 1$, then $d > 0$.*

Proof. Fix $(u, w) \in \mathcal{N}$. In view of (2.14) and (2.17), we get

$$\mathcal{J}(u, w) \ge \left(\frac{1}{2} - \frac{1}{c}\right)(\|\nabla u\|_2^2 + |\Delta w|_2^2), \tag{2.20}$$

where $c := \min\{p + 1, q + 1\} > 2$. Since $p \le 5$, it follows from (2.11), (2.17) and embedding inequalities that

$$\|\nabla u\|_2^2 + |\Delta w|_2^2 \le C_{p,q}(\|u\|_{p+1}^{p+1} + |w|_{q+1}^{q+1}) \le C(\|\nabla u\|_2^{p+1} + |\Delta w|_2^{q+1}),$$

which gives us

$$\|(u, w)\|_X^2 \le C(\|(u, w)\|_X^{p+1} + \|(u, w)\|_X^{q+1}). \tag{2.21}$$

Noting $(u, w) \ne (0, 0)$, we infer from (2.21) that

$$\|(u, w)\|_X^{p-1} + \|(u, w)\|_X^{q-1} \ge \frac{1}{C}.$$

Then, $\|(u, w)\|_X \ge s_0 > 0$, where s_0 is the unique positive root of the equation $s^{p-1} + s^{q-1} = \frac{1}{C}$. It follows from (2.20) that

$$\mathcal{J}(u, w) \ge \left(\frac{1}{2} - \frac{1}{c}\right) s_0^2, \quad \text{for all } (u, w) \in \mathcal{N},$$

completing the proof. □

We remark that the potential well $\mathcal{W}$ and the Nehari manifold $\mathcal{N}$ are disjoint sets, because of (2.18) and (2.19). That is,

$$\mathcal{W} \cap \mathcal{N} = \emptyset. \tag{2.22}$$

In other words, if $(u, w) \in X\backslash\{(0, 0)\}$ such that

$$\|\nabla u\|_2^2 + |\Delta w|_2^2 = (p + 1)\int_\Omega F(u)dx + (q + 1)\int_\Gamma H(w)d\Gamma,$$

then $(u, w) \notin \mathcal{W}$.

The potential well $\mathcal{W}$ can be decomposed into two parts: the “stable” part $\mathcal{W}_1$ and the “unstable” part $\mathcal{W}_2$:

$$\mathcal{W}_1 = \left\{ (u, w) \in \mathcal{W} : \|\nabla u\|_2^2 + |\Delta w|_2^2 > (p+1) \int_\Omega F(u) dx + (q+1) \int_\Gamma H(w) d\Gamma \right\} \cup \{(0,0)\},$$

$$\mathcal{W}_2 = \left\{ (u, w) \in \mathcal{W} : \|\nabla u\|_2^2 + |\Delta w|_2^2 < (p+1) \int_\Omega F(u) dx + (q+1) \int_\Gamma H(w) d\Gamma \right\}.$$

Clearly, $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. Moreover, because of (2.22), we see that

$$\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}.$$

In this paper, we show that if the initial data $(u_0, w_0) \in \mathcal{W}_1$ and the initial total energy $\mathcal{E}(0) < d$, then the weak solution of system (1.1) is global in time. Also, we prove the uniform decay rate of energy under additional assumptions. In another paper [14], we study the finite-time blow-up when the initial data resides in $\mathcal{W}_2$.

Using a similar argument as the proof of Lemma 2.7 in [19], the depth of the potential well d coincides with the mountain pass level. In particular,

$$d := \inf_{(u,w)\in\mathcal{N}} \mathcal{J}(u, w) = \inf_{(u,w)\in X\backslash\{(0,0)\}} \sup_{\lambda\geq 0} \mathcal{J}(\lambda(u, w)), \tag{2.23}$$

if the assumptions of Lemma 2.8 are valid. The minimax method and mountain pass theorem in the theory of calculus of variations can be found in the book by Rabinowitz [35].

2.4. A closed subset of $\mathcal{W}_1$

Here, we construct a closed subset of $\mathcal{W}_1$. If initial data belongs to such a subset, we are able to estimate energy decay rates in Section 4.

Throughout, we assume $1 < p \leq 5$ and $q > 1$. Also, Assumption 2.1 and Assumption 2.6 hold.

Thanks to the Sobolev embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ for $1 < p \leq 5$ and $H^2_0(\Gamma) \hookrightarrow L^{q+1}(\Gamma)$ for $q > 1$, we can define the best embedding constants:

$$K_1 := \sup_{u\in H^1_{\Gamma_0}(\Omega)\backslash\{0\}} \frac{\|u\|_{p+1}^{p+1}}{\|\nabla u\|_2^{p+1}}, \quad K_2 := \sup_{w\in H^2_0(\Gamma)\backslash\{0\}} \frac{|w|_{q+1}^{q+1}}{|\Delta w|_2^{q+1}}. \tag{2.24}$$

It follows from (2.11) and (2.24) that

$$\begin{aligned}
\mathcal{J}(u, w) &\geq \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - M(\|u\|_{p+1}^{p+1} + |w|_{q+1}^{q+1}) \\
&\geq \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - MK_1\|\nabla u\|_2^{p+1} - MK_2|\Delta w|_2^{q+1} \\
&\geq \frac{1}{2}\|(u, w)\|_X^2 - MK_1\|(u, w)\|_X^{p+1} - MK_2\|(u, w)\|_X^{q+1},
\end{aligned} \tag{2.25}$$

where $X = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$ and $\|(u, w)\|_X = (\|\nabla u\|_2^2 + |\Delta w|_2^2)^{1/2}$.

Inequality (2.25) can be expressed as:

$$\mathcal{J}(u,w) \geq \Lambda(\|(u,w)\|_X), \quad \text{for any } (u,v) \in X, \tag{2.26}$$

where the function $\Lambda(s)$ is given by:

$$\Lambda(s) := \frac{1}{2}s^2 - MK_1 s^{p+1} - MK_2 s^{q+1}. \tag{2.27}$$

In view of $p, q > 1$, then

$$\Lambda'(s) = s[1 - MK_1(p+1)s^{p-1} - MK_2(q+1)s^{q-1}],$$

has only one positive zero at s^*, where s^* satisfies

$$MK_1(p+1)(s^*)^{p-1} + MK_2(q+1)(s^*)^{q-1} = 1. \tag{2.28}$$

It is easy to see that $\Lambda(s)$ has a maximum value at s^* on $[0,\infty)$, i.e., $\sup_{s\in[0,\infty)} \Lambda(s) = \Lambda(s^*) > 0$.

Now, define

$$\tilde{\mathcal{W}}_1 := \{(u,w) \in X : \|(u,w)\|_X < s^*, \mathcal{J}(u,w) < \Lambda(s^*)\}. \tag{2.29}$$

We remark that $\tilde{\mathcal{W}}_1$ is not the trivial set $\{(0,0)\}$, since for any $(u,w) \in X$, there is a small real number $c > 0$ such that the scaler multiple $c(u,w)$ belong to $\tilde{\mathcal{W}}_1$.

Lemma 2.9. *$\tilde{\mathcal{W}}_1$ is a subset of $\mathcal{W}_1$.*

Proof. For $(u,w) \in X\backslash\{(0,0)\}$, we infer from (2.26) that $\mathcal{J}(\lambda(u,w)) \geq \Lambda(\lambda\|(u,w)\|_X)$ for all $\lambda \geq 0$. Then we have

$$\sup_{\lambda\geq 0} \mathcal{J}(\lambda(u,w)) \geq \sup_{\lambda\geq 0} \Lambda(\lambda\|(u,w)\|_X) = \sup_{s\in[0,\infty)} \Lambda(s) = \Lambda(s^*).$$

By (2.23), we have

$$\Lambda(s^*) \leq d. \tag{2.30}$$

Combining (2.11) and (2.24), and by using (2.28), we obtain that for any $(u,w) \in X\backslash\{(0,0)\}$ with $\|(u,w)\|_X < s^*$,

$$\begin{aligned}
&(p+1)\int_\Omega F(u)dx + (q+1)\int_\Gamma H(w)d\Gamma \\
&\leq (p+1)MK_1\|\nabla u\|_2^{p+1} + (q+1)MK_2|\Delta w|_2^{q+1} \\
&\leq \|(u,w)\|_X^2\Big[(p+1)MK_1\|(u,w)\|_X^{p-1} + (q+1)MK_2\|(u,w)\|_X^{q-1}\Big]
\end{aligned}$$

$$< \|(u,w)\|_X^2 \Big[(p+1)MK_1(s^*)^{p-1} + (q+1)MK_2(s^*)^{q-1} \Big]$$
$$= \|(u,w)\|_X^2 = \|\nabla u\|_2^2 + |\Delta w|_2^2. \tag{2.31}$$

Because of (2.29), (2.30) and (2.31), we conclude that $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$. □

For each sufficiently small $\delta > 0$, we define a closed subset of $\tilde{\mathcal{W}}_1$ by

$$\tilde{\mathcal{W}}_1^\delta := \{(u,w) \in X : \|(u,w)\|_X \leq s^* - \delta,\ \mathcal{J}(u,w) \leq \Lambda(s^* - \delta)\}. \tag{2.32}$$

$\tilde{\mathcal{W}}_1^\delta$ is a closed set because the space X is complete and $\mathcal{J}$ is continuous from X to $\mathbb{R}$. Clearly,

$$\tilde{\mathcal{W}}_1^\delta \subset \tilde{\mathcal{W}}_1 \subset \mathcal{W}_1.$$

In Section 4, we shall show the energy decay by assuming the initial data come from the closed set $\tilde{\mathcal{W}}_1^\delta$. Such a closed subset of $\mathcal{W}_1$ was also used in [19] by Guo and Rammaha to show the decay of energy for a system of coupled nonlinear wave equations. But, in paper [19] the closed set $\tilde{\mathcal{W}}_1^\delta$ was used in the lengthy compactness-uniqueness argument to absorb the lower-oder terms; whereas, in this manuscript we adjust the stabilization estimate by taking advantage of the closed set $\tilde{\mathcal{W}}_1^\delta$ so that the appearance of lower-order terms is prevented, and so our proof is concise.

2.5. *Total energy*

The kinetic energy of system (1.1) is given by $\frac{1}{2}(\|u_t(t)\|_2^2 + |w_t(t)|_2^2)$. Also, the potential energy is given by $\mathcal{J}(u(t), w(t))$, where the nonlinear function $\mathcal{J}$ is defined in (2.14). Naturally, we define the total energy $\mathcal{E}(t)$ as the summation of the kinetic energy and the potential energy, namely,

$$\begin{aligned}
\mathcal{E}(t) :&= \frac{1}{2}(\|u_t(t)\|_2^2 + |w_t(t)|_2^2) + \mathcal{J}(u(t), w(t)) \\
&= \frac{1}{2}\left(\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2\right) - \int_\Omega F(u)dx - \int_\Gamma H(w)d\Gamma \\
&= E(t) - \int_\Omega F(u)dx - \int_\Gamma H(w)d\Gamma,
\end{aligned} \tag{2.33}$$

where the quadratic energy $E(t)$ is defined in (2.7).

Using the notion of the total energy $\mathcal{E}(t)$, then the energy identity (2.8) can be rewritten in a simpler form. Indeed, since $\frac{d}{dt}F(u(t)) = f(u(t))u_t(t)$, we have $\int_0^t f(u)u_t d\tau = F(u(t)) - F(u_0)$. Therefore, the energy identity (2.8) is equivalent to

$$\mathcal{E}(t) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau = \mathcal{E}(0), \quad \text{for all } t \in [0,T), \tag{2.34}$$

where T is the maximal existence time. Taking the derivative with respect to t gives

$$\mathcal{E}'(t) + \int_{\Omega} g_1(u_t(t))u_t(t)\,dx + \int_{\Gamma} g_2(w_t(t))w_t(t)d\Gamma = 0, \quad \text{for all } t \in [0, T). \tag{2.35}$$

Since the frictional damping terms satisfy $g_1(s)s \geq 0$ and $g_2(s)s \geq 0$ for all $s \in \mathbb{R}$, we see from (2.35) that

$$\mathcal{E}'(t) \leq 0, \quad \text{for all } t \in [0, T). \tag{2.36}$$

Therefore, $\mathcal{E}(t)$ is non-increasing for all $t \in [0, T)$.

2.6. Main results

Our first result is the global existence of potential well solutions, provided the initial data belong to the set $\mathcal{W}_1$, which stands for the stable part of the potential well.

Theorem 2.10 *(Global existence of potential well solutions). Assume that Assumption 2.1 and Assumption 2.6 hold. Let $1 < p \leq 5$ and $q > 1$. Assume further $(u_0, w_0) \in \mathcal{W}_1$ and the total energy $\mathcal{E}(0) < d$. Then, system (1.1) admits a global weak solution (u, w). In addition, for any $t \geq 0$, the potential energy $\mathcal{J}(u(t), w(t))$, the total energy $\mathcal{E}(t)$ and the quadratic energy $E(t)$ satisfy*

$$\begin{cases} (i) & \mathcal{J}(u(t), w(t)) \leq \mathcal{E}(t) \leq \mathcal{E}(0) < d, \\ (ii) & (u(t), w(t)) \in \mathcal{W}_1, \\ (iii) & E(t) < \dfrac{cd}{c-2}, \\ (iv) & \dfrac{c-2}{c} E(t) \leq \mathcal{E}(t) \leq E(t), \end{cases}$$

where $c = \min\{p+1, q+1\} > 2$.

Remark 2.11. There are two global existence results: in Theorem 2.5, the weak solution is global provided the damping terms dominate the source terms, i.e., $m \geq p$ and $r \geq q$; whereas, in Theorem 2.10, the existence of global solutions is assured by the assumption that the initial data belong to $\mathcal{W}_1$ and initial energy is sufficiently small.

Remark 2.12. The uniqueness of weak solutions has been addressed by Theorem 2.5. In particular, with additional assumptions, the global solutions featured in Theorem 2.10 are unique. The additional assumptions for uniqueness have two options: either (i) assuming $u_0 \in L^{3(p-1)}(\Omega)$ and $m \geq 3p - 4$ when $p > 3$, or (ii) assuming Assumption 2.3 and $u_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$. Either of these two options of assumptions together with Assumption 2.1 lead to uniqueness of weak solutions. One can find the proof of uniqueness in paper [8].

Before stating the energy decay results, we give the following definition on the growth rates of a damping function g near the origin.

Definition 2.13. Let $g : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function such that

$$a_1|s|^{\gamma} \le |g(s)| \le a_2|s|^{\gamma}, \quad \forall\, |s| < 1,$$

where a_1, a_2 and γ are positive constants. If $\gamma = 1$, the function g is said to be *linearly bounded near the origin*. If $\gamma > 1$, g is called *superlinear near the origin*. If $0 < \gamma < 1$, g is called *sublinear near the origin*.

The following theorem establishes the uniform decay rates of energy. We assume monotone increasing functions g_1 and g_2 satisfy

$$c_1|s|^{\gamma_1} \le |g_1(s)| \le c_2|s|^{\gamma_1} \text{ and } c_3|s|^{\gamma_2} \le |g_2(s)| \le c_4|s|^{\gamma_2}, \quad \forall\, |s| < 1, \tag{2.37}$$

where $c_1, c_2, c_3, c_4, \gamma_1, \gamma_2$ are positive constants.

Theorem 2.14 *(Energy decay rates). Assume that Assumption 2.1 and Assumption 2.6 hold. Further assume that $1 < p < 5$, $k > 1$, and $u_0 \in L^{m+1}(\Omega)$. Also assume $(u_0, w_0) \in \tilde{\mathcal{W}}_1^{\delta}$ and $\mathcal{E}(0) \le \Lambda(s^* - \delta)$, for a sufficiently small $\delta > 0$. Moreover, assume $u \in L^{\infty}(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ if $m > 5$. Assume g_1 and g_2 satisfy (2.37) near the origin. Then the global solution of problem (1.1) furnished by Theorem 2.10 has the following decay rates.*

(i) If g_1 and g_2 are linearly bounded near the origin, then the total energy $\mathcal{E}(t)$ and the quadratic energy $E(t)$ decay to zero exponentially, namely, for any $t \ge 0$,

$$\frac{c-2}{c} E(t) \le \mathcal{E}(t) \le \frac{C\mathcal{E}(0)}{e^{at}}, \tag{2.38}$$

where C and a are positive constants independent of initial data, and $c = \min\{p+1, q+1\}$.

(ii) If at least one of g_1 and g_2 are either superlinear or sublinear near the origin, then the total energy $\mathcal{E}(t)$ and the quadratic energy $E(t)$ decay algebraically,

$$\frac{c-2}{c} E(t) \le \mathcal{E}(t) \le \frac{C(\mathcal{E}(0))}{(1+t)^b}, \tag{2.39}$$

where b is given by (4.10).

3. Global existence of potential well solutions

This section is devoted to proving the global existence of potential well solutions if the initial data are in the stable set $\mathcal{W}_1$. In particular, we justify Theorem 2.10 using the following argument.

Proof of Theorem 2.10. Let the initial data $(u_0, w_0) \in \mathcal{W}_1$ and $\mathcal{E}(0) < d$. The local well-posedness of a weak solution $(u(t), w(t))$ on $[0, T)$ is guaranteed by Theorem 2.5 from paper [8], where $[0, T)$ is the maximal interval of existence.

We first show $(u(t), w(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$, namely, if the initial data belong to $\mathcal{W}_1$, the solution trajectory $(u(t), w(t))$ belongs to $\mathcal{W}_1$ during the entire life spans of the solution.

Because of (2.36), we know $\mathcal{E}(t)$ is non-increasing as long as the solution exists. Therefore, $\mathcal{E}(t) \le \mathcal{E}(0) < d$ since we assume the initial total energy is less than the depth d of the potential well. Also, the potential energy $\mathcal{J}(u(t), w(t))$ is not larger than the total energy $\mathcal{E}(t)$. This gives us that for any $t \in [0, T)$,

$$\mathcal{J}(u(t), w(t)) \leq \mathcal{E}(t) \leq \mathcal{E}(0) < d. \tag{3.1}$$

Then part (i) is proved, and we obtain that $(u(t), w(t)) \in \mathcal{W}$ for all $t \in [0, T)$.

To prove $(u(t), w(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$, we argue by contradiction. We assume that there exists a time $t_1 \in (0, T)$ such that $(u(t_1), w(t_1)) \notin \mathcal{W}_1$. But $(u(t_1), w(t_1)) \in \mathcal{W}$. BY recalling $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, then it must be the case $(u(t_1), w(t_1)) \in \mathcal{W}_2$.

Recall $F'(\xi) = f(\xi)$ and $|f(\xi)| \leq (p+1)M|\xi|^p$ for any $\xi \in \mathbb{R}$ due to (2.12). Now, for any t, $t_0 \in [0, T)$, then by the mean value theorem, one has,

$$\begin{aligned}\int_\Omega |F(u(t)) - F(u(t_0))|dx &\leq C \int_\Omega (|u(t)|^p + |u(t_0)|^p)|u(t) - u(t_0)|dx \\ &\leq C(\|u(t)\|_{p+1}^p + \|u(t_0)\|_{p+1}^p)\|u(t) - u(t_0)\|_{p+1} \\ &\leq C(\|\nabla u(t)\|_2^p + \|\nabla u(t_0)\|_2^p)\|\nabla(u(t) - u(t_0))\|_2,\end{aligned} \tag{3.2}$$

where we have used the assumption that $p \leq 5$ and the embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^6(\Omega)$. By the continuity of $u(t)$, namely, $u \in C([0, T); H^1_{\Gamma_0}(\Omega))$, we can let t approach t_0 in (3.2) to conclude that

$$\int_\Omega F(u(t)) \to \int_\Omega F(u(t_0)), \quad \text{as} \quad t \to t_0,$$

which implies that the function $t \mapsto \int_\Omega F(u(t))dx$ is continuous on $[0, T)$.

Similarly, we can show that the function $t \mapsto \int_\Gamma H(w(t))d\Gamma$ is continuous on $[0, T)$, by using $w \in C([0, T); H^2_0(\Gamma))$.

Therefore, the mapping

$$t \mapsto \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2 - (p+1)\int_\Omega F(u(t))dx - (q+1)\int_\Gamma H(w(t))d\Gamma \tag{3.3}$$

is continuous.

In view of $(u(0), w(0)) \in \mathcal{W}_1$ and $(u(t_1), w(t_1)) \in \mathcal{W}_2$ as well as the continuity of the function in (3.3), the intermediate value theorem asserts that there exists a time $s \in (0, t_1)$ such that

$$\|\nabla u(s)\|_2^2 + |\Delta w(s)|_2^2 = (p+1)\int_\Omega F(u(s))dx + (q+1)\int_\Gamma H(w(s))d\Gamma. \tag{3.4}$$

Define t^* be the supremum of all $s \in (0, t_1)$ satisfying (3.4). Because of the continuity of the function in (3.3), we see that $t^* \in (0, t_1)$ satisfying (3.4), and $(u(t), w(t)) \in \mathcal{W}_2$ for any $t \in (t^*, t_1]$. We consider two cases:

Case 1. $(u(t^*), w(t^*)) \neq (0,0)$. Since (3.4) holds for t^*, then $(u(t^*), w(t^*)) \in \mathcal{N}$, by the definition of the Nehari manifold $\mathcal{N}$ in (2.17). Then, we obtain from (2.19) that $\mathcal{J}(u(t^*), w(t^*)) \geq d$, contradicting (3.1).

Case 2. $(u(t^*), w(t^*)) = (0,0)$. Note that $(u(t), w(t)) \in \mathcal{W}_2$ for any $t \in (t^*, t_1]$. We conclude from the definition of the set $\mathcal{W}_2$ and (2.11) that for any $t \in (t^*, t_1]$,

$$\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2 < C(\|u(t)\|_{p+1}^{p+1} + |w(t)|_{q+1}^{q+1}) \leq C(\|\nabla u(t)\|_2^{p+1} + |\Delta w(t)|_2^{q+1}),$$

because $p \leq 5$. This implies that

$$\|(u(t), w(t))\|_X^2 < C(\|(u(t), w(t))\|_X^{p+1} + \|(u(t), w(t))\|_X^{q+1}), \text{ for all } t \in (t^*, t_1]. \tag{3.5}$$

Since $(0,0)$ does not belong to $\mathcal{W}_2$, then $(u(t), w(t)) \neq (0,0)$ for any $t \in (t^*, t_1]$. Then, we can divide both sides of (3.5) by $\|(u(t), w(t))\|_X^2$ to obtain

$$\|(u(t), w(t))\|_X^{p-1} + \|(u(t), w(t))\|_X^{q-1} > \frac{1}{C}.$$

This yields $\|(u(t), w(t))\|_X > s_0$, for any $t \in (t^*, t_1]$, where $s_0 > 0$ is the unique positive solution of $s^{p-1} + s^{q-1} = \frac{1}{C}$, where $p, q > 1$. Since the weak solution $(u(t), w(t))$ is continuous from $[0, T)$ to X, one has $\|(u(t^*), w(t^*))\|_X \geq s_0 > 0$. This contradicts the assumption $(u(t^*), w(t^*)) = (0,0)$. Therefore $(u(t), w(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$. Hence, claim (ii) is proved.

In the following, we prove that the weak solution $(u(t), w(t))$ on $[0, T)$ is global in time, i.e., the maximum lifespan $T = \infty$. To this end, we need to prove that the quadratic energy $E(t)$ has a uniform bound independent of time.

By (3.1), the potential energy $\mathcal{J}(u(t), w(t)) < d$ for all $t \in [0, T)$, i.e.,

$$d > \mathcal{J}((u(t), w(t))) > \frac{1}{2}(\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2) - \int_\Omega F(u(t))dx - \int_\Gamma H(w(t))d\Gamma. \tag{3.6}$$

Since we have proved that $(u(t), w(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$, then

$$\int_\Omega F(u(t))dx + \int_\Gamma H(w(t))d\Gamma \leq \frac{1}{c}(\|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2), \tag{3.7}$$

where $c = \min\{p+1, q+1\} > 2$. It follows from (3.6) and (3.7) that for any $t \in [0, T)$,

$$\int_\Omega F(u(t))dx + \int_\Gamma H(w(t))d\Gamma < \frac{2d}{c-2}. \tag{3.8}$$

Substituting (3.8) into (2.34), we get that for any $t \in [0, T)$,

$$
\begin{aligned}
E(t) &+ \int_0^t \int_\Omega g_1(u_t)u_t\,dx\,d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t\,d\Gamma\,d\tau \\
&= \mathcal{E}(0) + \int_\Omega F(u(t))dx + \int_\Gamma H(w(t))d\Gamma \\
&< d + \frac{2d}{c-2} = \frac{cd}{c-2}.
\end{aligned}
$$

Since $g_1(s)s \geq 0$ and $g_2(s)s \geq 0$ for all $s \in \mathbb{R}$, we obtain

$$
E(t) < \frac{cd}{c-2}, \quad \text{for all } t \in [0, T), \tag{3.9}
$$

proving claim (iii).

Recall Theorem 2.5 states that the local well-posedness of weak solutions defined on $[0, T_0]$ where T_0 depends on the initial quadratic energy $E(0)$. Due to (3.9), $E(T_0)$ and $E(0)$ have the same upper bound. So we can extend the local solution from the time T_0 to $2T_0$. By iterating this procedure, one can obtain a global weak solution defined on $[0, \infty)$. That is to say, the maximum lifespan $T = \infty$.

To show (iv), we observe from (2.33) that $\mathcal{E}(t) \leq E(t)$ for all $t \in [0, \infty)$ since F and H are nonnegative functions. Also, by (2.33) and (3.7), we obtain

$$
\mathcal{E}(t) \geq \frac{1}{2}(\|u_t\|_2^2 + |w_t|_2^2) + \left(\frac{1}{2} - \frac{1}{c}\right)(\|\nabla u\|_2^2 + |\Delta w|_2^2) \geq \frac{c-2}{c}E(t).
$$

The proof for Theorem 2.10 is complete. □

Remark 3.1. The continuity of the solution (u, w) mapping from $[0, T)$ to $X = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$ is critical for the argument above, and essential for the validity of the entire paper. For instance, the implementation of the intermediate value theorem in the above proof depends on the continuity of the solution. Such regularity (namely, $u \in C([0, T); H^1_{\Gamma_0}(\Omega))$ and $w \in C([0, T); H^2(\Gamma))$) is guaranteed by the local well-posedness result (Theorem 2.5), proved by Becklin and Rammaha in [8]. The method to prove the local existence for system (1.1) in [8] consists of the theory of monotone operators and nonlinear semi-groups. Specifically, using Kato's Theorem (see, e.g. [37]), the system has a solution $(u, w) \in W^{1,\infty}(0, T; X)$ if the source terms are subcritical. Then, the extension to supercritical source terms concludes that $(u, w) \in C([0, T); X)$. On the other hand, in another paper [7], Becklin and Rammaha studied a related model with restoring source terms but no damping terms, and by using Galerkin method, the local existence of weak solutions was shown but the solutions have only weak continuity in time.

We end this section by giving a proof of this following lemma, which states that if the initial data belong to $\tilde{\mathcal{W}}_1^\delta$ (the closed subset of $\mathcal{W}_1$ constructed in subsection 2.4) and if the initial total energy is sufficiently small, then the solution remains in $\tilde{\mathcal{W}}_1^\delta$ for all time.

Recall the function $\Lambda(s)$ is defined in (2.27) and s^* is the location of the maximum of $\Lambda(s)$ on $\mathbb{R}^+$. Also, recall the set $\tilde{\mathcal{W}}_1^\delta$ is defined in (2.32).

Lemma 3.2. *Suppose Assumption 2.1 and Assumption 2.6 are valid. Let $1 < p \le 5$, $q > 1$, and $\delta > 0$ is sufficiently small. Assume $\mathcal{E}(0) \le \Lambda(s^* - \delta)$ and $(u_0, w_0) \in \tilde{\mathcal{W}}_1^\delta$. Then system (1.1) admits a global solution (u, v) satisfying $(u(t), w(t)) \in \tilde{\mathcal{W}}_1^\delta$ for all $t \ge 0$.*

Proof. Recall that, in subsection 2.4, we have shown that the function $\Lambda(s)$ defined in (2.27) attains it maximum at $s = s^*$ over $\mathbb{R}^+$, and $\Lambda(s^*) \le d$ due to (2.30). Since $\Lambda(t)$ is strictly increasing on $(0, s^*)$, we see that $\mathcal{E}(0) \le \Lambda(s^* - \delta) < d$. Also, since $(u_0, w_0) \in \tilde{\mathcal{W}}_1^\delta \subset \mathcal{W}_1$, then thanks to Theorem 2.10, there exists a global solution (u, v) with $\mathcal{J}(u(t), w(t)) \le \mathcal{E}(0) \le \Lambda(s^* - \delta)$ for all $t \ge 0$. It remains to show that $\|(u(t), w(t))\|_X \le s^* - \delta$ for all $t \ge 0$. Since $\|(u_0, w_0)\|_X \le s^* - \delta$ and $(u, w) \in C(\mathbb{R}^+, X)$, we assume to the contrary that there exists $t_1 > 0$ such that $\|(u(t_1), w(t_1))\|_X = s^* - \delta + \delta_0$ for some $\delta_0 \in (0, \delta)$. By (2.26), we have $\mathcal{J}(u(t_1), w(t_1)) \ge \Lambda(s^* - \delta + \delta_0) > \Lambda(s^* - \delta)$, because $\Lambda(t)$ is strictly increasing on $(0, s^*)$. This contradicts $\mathcal{J}(u(t), w(t)) \le \Lambda(s^* - \delta)$ for any $t \ge 0$. □

4. Energy decay rates

This section is devoted to proving Theorem 2.14, namely, the uniform energy decay rates of potential well solutions.

First we remark that, if the solution decays, then at large time, the solution becomes "small", so the behavior of damping terms near the origin determines the decay rates of solutions.

In what follows, we introduce some concave functions that capture the growth rates of damping terms near the origin.

4.1. Concave functions that reflect the behavior of damping terms near the origin

Let $\phi_i : [0, \infty) \to [0, \infty)$ be concave, increasing, continuous functions vanishing at the origin, such that, for $i = 1, 2$,

$$\phi_i(g_i(s)s) \ge |g_i(s)|^2 + s^2, \quad \text{for } |s| < 1. \tag{4.1}$$

Also, we define a function $\Phi : [0, \infty) \to [0, \infty)$ by

$$\Phi(s) := \phi_1(s) + \phi_2(s) + s, \quad s \ge 0. \tag{4.2}$$

Note the function Φ is also concave, increasing, continuous and vanishing at the origin.

Now, we show that the concave functions ϕ_i ($i = 1, 2$) satisfying (4.1) can always be constructed. Indeed, recall that g_1 and g_2 are continuous monotone increasing functions passing the origin. If g_1 and g_2 are bounded above and below by linear or superlinear functions near the origin, i.e.,

$$c_1|s|^m \le |g_1(s)| \le c_2|s|^m, \quad c_3|s|^r \le |g_2(s)| \le c_4|s|^r, \quad \forall\, |s| < 1, \tag{4.3}$$

where $m, r \ge 1$ and $c_i > 0$, $i = 1, 2, 3, 4$, then we choose

$$\phi_1(s) = (c_1)^{-\frac{2}{m+1}}(1 + c_2^2)s^{\frac{2}{m+1}}, \quad \phi_2(s) = (c_3)^{-\frac{2}{r+1}}(1 + c_4^2)s^{\frac{2}{r+1}}. \tag{4.4}$$

To check that (4.4) satisfies (4.1), we calculate directly:

$$\phi_1(g_1(s)s) = (c_1)^{-\frac{2}{m+1}}(1+c_2^2)(g_1(s)s)^{\frac{2}{m+1}} \geq c_1^{-\frac{2}{m+1}}(1+c_2^2)(c_1|s|^{m+1})^{\frac{2}{m+1}}$$
$$= (1+c_2^2)s^2 \geq s^2 + (c_2|s|^m)^2 \geq s^2 + |g_1(s)|^2, \text{ for all } |s| < 1.$$

In particular, if g_1 and g_2 are both linearly bounded near the origin, then the explicit functions ϕ_1 and ϕ_2 given in (4.4) are both linear functions.

On the other hand, if g_1 and g_2 are bounded by sublinear functions near the origin, that is,

$$c_1|s|^{\kappa_1} \leq |g_1(s)| \leq c_2|s|^{\kappa_1}, \quad c_3|s|^{\kappa_2} \leq |g_2(s)| \leq c_4|s|^{\kappa_2}, \quad \text{for all } |s| < 1, \tag{4.5}$$

where $0 < \kappa_1, \kappa_2 < 1$ and $c_i > 0$ $(i = 1, 2, 3, 4)$. In this case, we can select

$$\phi_1(s) = (c_1)^{-\frac{2\kappa_1}{\kappa_1+1}}(1+c_2^2)s^{\frac{2\kappa_1}{\kappa_1+1}}, \quad \phi_2(s) = (c_3)^{-\frac{2\kappa_2}{\kappa_2+1}}(1+c_4^2)s^{\frac{2\kappa_2}{\kappa_2+1}}. \tag{4.6}$$

From (4.4) and (4.6), we know that one can always construct ϕ_1 and ϕ_2 in the form

$$\phi_1(s) = C_1 s^{\nu_1} \text{ and } \phi_2(s) = C_2 s^{\nu_2}, \tag{4.7}$$

for some constants C_1 and C_2, where

$$\nu_1 = \frac{2}{m+1} \text{ or } \frac{2\kappa_1}{\kappa_1+1}, \quad \nu_2 = \frac{2}{r+1} \text{ or } \frac{2\kappa_2}{\kappa_2+1}, \tag{4.8}$$

depending on the growth rates of g_1 and g_2 near the origin, which are specified in (4.3) and (4.5).

Define

$$j := \max\left\{\frac{1}{\nu_1}, \frac{1}{\nu_2}\right\}. \tag{4.9}$$

Note that $j > 1$ if at least one of g_1 and g_2 are either superlinear or sublinear near the origin, and in this case we put

$$b := (j-1)^{-1} > 0. \tag{4.10}$$

4.2. *A stabilization estimate*

For convenience, we put:

$$D(t) := \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau. \tag{4.11}$$

Since $g_1(s)s \geq 0$ and $g_2(s)s \geq 0$, then $D(t) \geq 0$. Using this notation, the energy identity (2.34) can be written in the concise form:

$$\mathcal{E}(t) + D(t) = \mathcal{E}(0). \tag{4.12}$$

From (4.12), we see that the behavior of damping terms determines the decay rates of the total energy $\mathcal{E}(t)$.

Define

$$T_0 := \max\left\{1, \frac{1}{|\Omega|}, \frac{1}{|\Gamma|}, \frac{8cc_0}{c-2}\right\}, \tag{4.13}$$

where $c = \min\{p+1, q+1\} > 2$ and $c_0 > 0$ is defined in (4.19).

Lemma 4.1. *Suppose that Assumption 2.1 and Assumption 2.6 hold. Assume that $1 < p < 5$, $k > 1$, and $u_0 \in L^{m+1}(\Omega)$. Also assume $(u_0, w_0) \in \tilde{\mathcal{W}}_1^\delta$ and $\mathcal{E}(0) \le \Lambda(s^* - \delta)$, for a sufficiently small $\delta > 0$. In addition, suppose $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ if $m > 5$. Then the global solution of system (1.1) furnished by Theorem 2.10 satisfies for all $T \ge T_0$,*

$$\mathcal{E}(T) \le \tilde{C}\Phi(D(T)), \tag{4.14}$$

where T_0 is given in (4.13), Φ is defined in (4.2), and $\tilde{C} > 0$ defined in (4.53) is independent of T.

Proof. Throughout the proof, we assume $T \ge T_0$, where T_0 is given in (4.13). By the regularity of weak solutions specified in Definition 2.4, we know that $u_t \in L^{m+1}(\Omega \times (0, T))$. Since we assume $u_0 \in L^{m+1}(\Omega)$, then the fundamental theorem of Calculus implies

$$\begin{aligned}\int_0^T\int_\Omega |u|^{m+1} dxdt &= \int_0^T\int_\Omega \left|\int_0^t u_t(\tau)d\tau + u_0\right|^{m+1} dxdt \\ &\le C(T^{m+1}\|u_t(t)\|^{m+1}_{L^{m+1}(\Omega\times(0,T))} + T\|u_0\|^{m+1}_{m+1}) < \infty. \end{aligned}\tag{4.15}$$

This implies $u \in L^{m+1}(\Omega \times (0, T))$. We can use the same argument to get $w \in L^{r+1}(\Gamma \times (0, T))$. In this situation, we can replace ϕ by u in (2.5) and ψ by w in (2.6); then adding the results yield

$$\begin{aligned}&\int_\Omega u_t u dx\Big|_0^T + \int_\Gamma (w_t w + \gamma u w)d\Gamma\Big|_0^T - \int_0^T (\|u_t\|_2^2 + |w_t|_2^2)dt + \int_0^T (\|\nabla u\|_2^2 + |\Delta w|_2^2)dt \\ &- 2\int_0^T\int_\Gamma \gamma u w_t d\Gamma dt + \int_0^T\int_\Omega g_1(u_t)u dxdt + \int_0^T\int_\Gamma g_2(w_t)w d\Gamma dt \\ &= \int_0^T\int_\Omega f(u)u dxdt + \int_0^T\int_\Gamma h(w)w d\Gamma dt. \end{aligned}\tag{4.16}$$

Multiply equality (4.16) by 1/2, and use (2.7) and (2.9), we obtain

$$\int_0^T E(t)dt = \underbrace{\int_0^T (\|u_t\|_2^2 + |w_t|_2^2)dt}_{=I_3} \underbrace{-\frac{1}{2}\int_\Omega u_t u dx\Big|_0^T - \frac{1}{2}\int_\Gamma (w_t w + \gamma u w)d\Gamma\Big|_0^T}_{=I_1}$$
$$\underbrace{+\int_0^T\int_\Gamma \gamma u w_t d\Gamma dt}_{=I_4} \underbrace{-\frac{1}{2}\int_0^T\int_\Omega g_1(u_t)u dx dt}_{=I_6} \underbrace{-\frac{1}{2}\int_0^T\int_\Gamma g_2(w_t) w d\Gamma dt}_{=I_5}$$
$$\underbrace{+\frac{(p+1)}{2}\int_0^T\int_\Omega F(u)dxdt + \frac{q+1}{2}\int_0^T\int_\Gamma H(w)d\Gamma dt}_{=I_2}. \tag{4.17}$$

In the sequel, we will estimate terms I_i $(i = 1, ..., 6)$ one by one. □

4.2.1. Estimate for I_1

First, by using Cauchy-Schwarz inequality and Sobolev embedding theorem, we get

$$\begin{aligned}\left|\int_\Omega u_t u dx + \int_\Gamma (w_t w + \gamma u w)d\Gamma\right| &\le \frac{1}{2}(\|u_t\|_2^2 + |w_t|_2^2 + \|u\|_2^2 + 2|w|_2^2 + |\gamma u|_2^2)\\ &\le \frac{1}{2}\Big[\|u_t\|_2^2 + |w_t|_2^2 + (c_1^* + c_*)\|\nabla u\|_2^2 + 2c_2^*|\Delta w|_2^2\Big]\\ &\le c_0 E(t),\end{aligned} \tag{4.18}$$

where

$$c_0 = \max\{1, (c_1^* + c_*), 2c_2^*\}, \tag{4.19}$$

and $c_1^* > 0$ is the embedding constant of $\|u\|_2^2 \le c_1^*\|\nabla u\|_2^2$, $c_2^* > 0$ is the embedding constant of $|w|_2^2 \le c_2^*|\Delta w|_2^2$ and $c_* > 0$ is the embedding constant of $|\gamma u|_2^2 \le c_*\|\nabla u\|_2^2$.

It follows from claim (iv) of Theorem 2.10, (4.18) and (4.12) that

$$I_1 \le \frac{c_0}{2}(E(T) + E(0)) \le \frac{cc_0}{2(c-2)}(\mathcal{E}(T) + \mathcal{E}(0)) \le \frac{cc_0}{2(c-2)}(2\mathcal{E}(T) + D(T)). \tag{4.20}$$

4.2.2. Estimate for I_2

It follows from (2.11), (2.13) and (2.24) that

$$I_2 \le \frac{1}{2}M(p+1)\int_0^T \|u\|_{p+1}^{p+1}dt + \frac{1}{2}M(q+1)\int_0^T |w|_{q+1}^{q+1}dt$$

$$\leq \frac{1}{2}M(p+1)K_1\int_0^T \|\nabla u\|_2^{p+1}dt + \frac{1}{2}M(q+1)K_2\int_0^T |\Delta w|_2^{q+1}dt$$

$$\leq \frac{1}{2}M(p+1)K_1\int_0^T \|(u,w)\|_X^{p+1}dt + \frac{1}{2}M(q+1)K_2\|(u,w)\|_X^{q+1}dt$$

$$= \frac{1}{2}\int_0^T \|(u,w)\|_X^2[M(p+1)K_1\|(u,w)\|_X^{p-1} + M(q+1)K_2\|(u,w)\|_X^{q-1}]dt. \quad (4.21)$$

Since $(u_0, w_0) \in \tilde{\mathcal{W}}_1^\delta$ and $\mathcal{E}(0) \leq \Lambda(s^* - \delta)$ and thanks to Lemma 3.2, we have $(u(t), w(t)) \in \tilde{\mathcal{W}}_1^\delta$ for all $t \geq 0$. Then due to the definition of $\tilde{\mathcal{W}}_1^\delta$, i.e., formula (2.32), we know that

$$\|(u(t), w(t))\|_X \leq s^* - \delta, \text{ for all } t \geq 0. \quad (4.22)$$

Then, we obtain from (4.21) and (4.22) that

$$I_2 \leq \frac{1}{2}\int_0^T \|(u,w)\|_X^2[M(p+1)K_1(s^*-\delta)^{p-1} + M(q+1)K_2(s^*-\delta)^{q-1}]dt$$

$$= \frac{1}{2}\xi\int_0^T \|(u,w)\|_X^2 \leq \xi\int_0^T E(t)dt, \quad (4.23)$$

where the constant ξ is defined as

$$\xi := M(p+1)K_1(s^*-\delta)^{p-1} + M(q+1)K_2(s^*-\delta)^{q-1} < 1. \quad (4.24)$$

The fact that $\xi < 1$ is because of (2.28).

In sum, we conclude that there exists a constant $0 < \xi < 1$ such that

$$I_2 \leq \xi\int_0^T E(t)dt. \quad (4.25)$$

We stress that the fact that ξ is strictly less than 1 is crucial for our argument. Because, the right-hand side of (4.25) can be completely absorbed by the term $\int_0^T E(t)dt$ on the left-hand side of (4.17). This makes the proof concise because there are no lower-order terms appearing in the stabilization estimate.

4.2.3. Estimate for I_3

We define

$$A_\Omega := \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| < 1\}, \tag{4.26}$$

and

$$B_\Omega := \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| \geq 1\}. \tag{4.27}$$

From Assumption 2.1, we infer that

$$\alpha|s|^2 \leq \alpha|s|^{m+1} \leq g_1(s)s, \quad \forall\, |s| \geq 1,$$

since $m \geq 1$.

Now we use the concave functions ϕ_1 and ϕ_2 constructed in subsection 4.1. Recall that ϕ_1 and ϕ_2 are related to the growth rates of g_1 and g_2 near the origin, respectively.

By (4.1) and noting that ϕ_1 maps $[0,\infty)$ to $[0,\infty)$, we derive that

$$\begin{aligned} \int_0^T \|u_t(t)\|_2^2 dx &= \int_{A_\Omega} |u_t|^2 dxdt + \int_{B_\Omega} |u_t|^2 dxdt \\ &\leq \int_{A_\Omega} \phi_1(g_1(u_t)u_t) dxdt + \frac{1}{\alpha} \int_{B_\Omega} g_1(u_t)u_t dxdt \\ &\leq \int_0^T \int_\Omega \phi_1(g_1(u_t)u_t) dxdt + \frac{1}{\alpha} \int_0^T \int_\Omega g_1(u_t)u_t dxdt. \end{aligned} \tag{4.28}$$

Since ϕ_1 is concave, we can use Jensen's inequality to obtain

$$\begin{aligned} \frac{1}{T|\Omega|} \int_0^T \int_\Omega \phi_1(g_1(u_t)u_t) dxdt &\leq \phi_1\left(\frac{1}{T|\Omega|} \int_0^T \int_\Omega g_1(u_t)u_t dxdt \right) \\ &\leq \phi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt \right), \end{aligned} \tag{4.29}$$

where we have used the fact that ϕ_1 is increasing and $T|\Omega| \geq 1$ because $T \geq T_0 \geq \frac{1}{|\Omega|}$ from (4.13).

Combining (4.28) and (4.29) yields

$$\int_0^T \|u_t(t)\|_2^2 dx \leq T|\Omega|\phi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt \right) + \frac{1}{\alpha} \int_0^T \int_\Omega g_1(u_t)u_t dxdt. \tag{4.30}$$

In the same manner, we can show

$$\int_0^T |w_t(t)|_2^2 dt \le T|\Gamma|\phi_2\left(\int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt\right) + \frac{1}{\alpha}\int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt. \tag{4.31}$$

Then it follows from (4.30) and (4.31) that

$$\begin{aligned} I_3 \le\ & T|\Omega|\phi_1\left(\int_0^T\int_\Omega g_1(u_t)u_t dxdt\right) + T|\Gamma|\phi_2\left(\int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt\right) \\ & + \frac{1}{\alpha}\left[\int_0^T\int_\Omega g_1(u_t)u_t dxdt + \int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt\right]. \end{aligned} \tag{4.32}$$

4.2.4. Estimate for I_4

By using Cauchy-Schwarz inequality, the embedding $|\gamma u|_2^2 \le c_*\|\nabla u\|_2^2$, and Young's inequality, we obtain

$$I_4 \le \int_0^T |\gamma u|_2|w_t|_2 dt \le c_*^{1/2}\int_0^T \|\nabla u\|_2|w_t|_2 dt \le \frac{\varepsilon}{2}\int_0^T \|\nabla u\|_2^2 dt + C_\varepsilon\int_0^T |w_t|_2^2 dt,$$

and together with (4.31), we obtain

$$I_4 \le \varepsilon\int_0^T E(t)dt + T|\Gamma|C_\varepsilon\phi_2\left(\int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt\right) + \frac{C_\varepsilon}{\alpha}\int_0^T\int_\Gamma g_2(w_t)w_t d\Gamma dt. \tag{4.33}$$

4.2.5. Estimate for I_5

To estimate I_5, as A_Ω and B_Ω, we define

$$A_\Gamma := \{(x,t) \in \Gamma \times (0,T) : |w_t(x,t)| < 1\},$$

and

$$B_\Gamma := \{(x,t) \in \Gamma \times (0,T) : |w_t(x,t)| \ge 1\}.$$

By using Hölder's inequality, Young's inequality and the definition of $E(t)$, we get that for any $\varepsilon > 0$,

$$\int_0^T\int_\Gamma |g_2(w_t)w| d\Gamma dt = \int_{A_\Gamma} |g_2(w_t)w| d\Gamma dt + \int_{B_\Gamma} |g_2(w_t)w| d\Gamma dt$$

$$\leq \left(\int_0^T |w|_2^2 dt\right)^{\frac{1}{2}} \left(\int_{A_\Gamma} |g_2(w_t)|^2 d\Gamma dt\right)^{\frac{1}{2}} + \int_{B_\Gamma} |g_2(w_t)w| d\Gamma dt$$

$$\leq \varepsilon \int_0^T E(t)dt + C_\varepsilon \int_{A_\Gamma} |g_2(w_t)|^2 d\Gamma dt + \int_{B_\Gamma} |g_2(w_t)w| d\Gamma dt, \tag{4.34}$$

where we have used the Poincaré inequality $|w|_2 \leq C|\Delta w|_2^2 \leq CE(t)$.

Since $T \geq T_0 \geq \frac{1}{|\Gamma|}$ from (4.13), we have $T|\Gamma| \geq 1$. Also, recall the function $\phi_2 : [0, \infty) \to [0, \infty)$ is concave. Then, we can use Jensen's inequality and (4.1) to deduce

$$\int_{A_\Gamma} |g_2(w_t)|^2 d\Gamma dt \leq \int_{A_\Gamma} \phi_2(g_2(w_t)w_t) d\Gamma dt \leq T|\Gamma|\phi_2\left(\int_0^T \int_\Gamma g_2(w_t)w_t d\Gamma dt\right). \tag{4.35}$$

Recalling Assumption 2.1, we have $|g_2(s)| \leq \beta|s|^r$ for all $|s| \geq 1$. Then Hölder's inequality implies

$$\int_{B_\Gamma} |g_2(w_t)w| d\Gamma dt \leq \left(\int_{B_\Gamma} |w|^{r+1} d\Gamma dt\right)^{\frac{1}{r+1}} \left(\int_{B_\Gamma} |g_2(w_t)|^{\frac{r+1}{r}} d\Gamma dt\right)^{\frac{r}{r+1}}$$

$$\leq \left(\int_0^T |w|_{r+1}^{r+1} dt\right)^{\frac{1}{r+1}} \left(\int_{B_\Gamma} |g_2(w_t)||g_2(w_t)|^{\frac{1}{r}} d\Gamma dt\right)^{\frac{r}{r+1}}$$

$$\leq \beta^{\frac{1}{r+1}} \left(\int_0^T |w|_{r+1}^{r+1} dt\right)^{\frac{1}{r+1}} \left(\int_{B_\Gamma} |g_2(w_t)|w_t d\Gamma dt\right)^{\frac{r}{r+1}}. \tag{4.36}$$

Recall that the claim (iii) of Theorem 2.10 tells us $E(t) < \frac{cd}{c-2}$ for all $t \geq 0$. Also, Sobolev embedding shows $|w|_{r+1} \leq C|\Delta w|_2$. Hence,

$$\int_0^T |w|_{r+1}^{r+1} dt \leq C \int_0^T |\Delta w|_2^{r+1} dt \leq C \int_0^T E^{\frac{r+1}{2}}(t) dt \leq C(c, d, r) \int_0^T E(t) dt, \tag{4.37}$$

since $r \geq 1$.

Combining (4.36) and (4.37), and using Young's inequality, we obtain that for any $\varepsilon > 0$,

$$\int_{B_\Gamma} |g_2(w_t)w| d\Gamma dt \leq C \left(\int_0^T E(t) dt\right)^{\frac{1}{r+1}} \left(\int_0^T \int_\Gamma |g_2(w_t)w_t| d\Gamma dt\right)^{\frac{r}{r+1}}$$

$$\leq \varepsilon \int_0^T E(t)dt + C_\varepsilon \int_0^T \int_\Gamma |g_2(w_t)w_t| d\Gamma dt. \tag{4.38}$$

Substituting (4.35) and (4.38) into (4.34), we get for any $\varepsilon > 0$,

$$I_5 = \frac{1}{2}\int_0^T \int_\Gamma |g_2(w_t)w| d\Gamma dt \leq \varepsilon \int_0^T E(t)dt + C_\varepsilon T|\Gamma|\phi_2\left(\int_0^T \int_\Gamma g_2(w_t)w_t d\Gamma dt\right)$$
$$+ C_\varepsilon \int_0^T \int_\Gamma g_2(w_t)w_t d\Gamma dt. \tag{4.39}$$

4.2.6. Estimate for I_6

Recall the sets A_Ω and B_Ω are defined in (4.26)-(4.27). Using Hölder's inequality and Young's inequality, we obtain

$$\int_0^T \int_\Omega |g_1(u_t)u| dxdt = \int_{A_\Omega} |g_1(u_t)u| dxdt + \int_{B_\Omega} |g_1(u_t)u| dxdt$$
$$\leq \left(\int_0^T \|u\|_2^2 dt\right)^{\frac{1}{2}} \left(\int_{A_\Omega} |g_1(u_t)|^2 dxdt\right)^{\frac{1}{2}} + \int_{B_\Omega} |g_1(u_t)u| dxdt$$
$$\leq \varepsilon \int_0^T E(t)dt + C_\varepsilon \int_{A_\Omega} |g_1(u_t)|^2 dxdt + \int_{B_\Omega} |g_1(u_t)u| dxdt. \tag{4.40}$$

for any $\varepsilon > 0$.

Using (4.1) and Jensen's inequality, we obtain

$$\int_{A_\Omega} |g_1(u_t)|^2 dxdt \leq \int_{A_\Omega} \phi_1(g_1(u_t)u_t) dxdt \leq T|\Omega|\phi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right), \tag{4.41}$$

due to $T|\Omega| \geq 1$.

We consider two cases to estimate the last term on the right-hand side of (4.40).

Case 1. $m \leq 5$.

It follows from Assumption 2.1 that $|g_1(s)| \leq \beta|s|^m \leq \beta|s|^5$ for $|s| \geq 1$. Then we apply Hölder's inequality to deduce

$$\int_{B_\Omega} |g_1(u_t)u| dxdt \leq \left(\int_{B_\Omega} |u|^6 dxdt\right)^{\frac{1}{6}} \left(\int_{B_\Omega} |g_1(u_t)|^{\frac{6}{5}} dxdt\right)^{\frac{5}{6}}$$

$$\leq \left(\int_0^T \|u\|_6^6 dt\right)^{\frac{1}{6}} \left(\int_{B_\Omega} |g_1(u_t)||g_1(u_t)|^{\frac{1}{5}} dxdt\right)^{\frac{5}{6}}$$

$$\leq \beta^{\frac{1}{6}} \left(\int_0^T \|u\|_6^6 dt\right)^{\frac{1}{6}} \left(\int_{B_\Omega} |g_1(u_t)||u_t| dxdt\right)^{\frac{5}{6}},$$

which along with

$$\int_0^T \|u\|_6^6 dt \leq C \int_0^T \|\nabla u\|_2^6 dt \leq C \int_0^T E^3(t) dt \leq C(c,d) \int_0^T E(t) dt,$$

yields that

$$\int_{B_\Omega} |g_1(u_t)u| dxdt \leq C \left(\int_0^T E(t) dt\right)^{\frac{1}{6}} \left(\int_{B_\Omega} g_1(u_t)u_t dxdt\right)^{\frac{5}{6}}$$

$$\leq \varepsilon \int_0^T E(t) dt + C_\varepsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \tag{4.42}$$

for any $\varepsilon > 0$.

Combining (4.41) and (4.42) with (4.40), we obtain that for any $\varepsilon > 0$,

$$I_6 = \frac{1}{2} \int_0^T \int_\Omega |g_1(u_t)u| dxdt \leq \varepsilon \int_0^T E(t) dt + C_\varepsilon T|\Omega|\phi_1 \left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right)$$

$$+ C_\varepsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \quad \text{if } m \leq 5. \tag{4.43}$$

Case 2. $m > 5$.

In this case, we assume an extra regularity on u, namely, $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$. Hölder's inequality implies

$$\int_{B_\Omega} |g_1(u_t)u| dxdt \leq \left(\int_{B_\Omega} |g_1(u_t)|^{\frac{m+1}{m}} dxdt\right)^{\frac{m}{m+1}} \left(\int_{B_\Omega} |u|^{m+1} dxdt\right)^{\frac{1}{m+1}}. \tag{4.44}$$

Since $|g_1(s)| \leq \beta|s|^m$ for $|s| \geq 1$, we have

$$\int_{B_\Omega} |g_1(u_t)|^{\frac{m+1}{m}} dxdt = \int_{B_\Omega} |g_1(u_t)||g_1(u_t)|^{\frac{1}{m}} dxdt \le \beta^{\frac{1}{m}} \int_{B_\Omega} g_1(u_t)u_t dxdt. \tag{4.45}$$

Moreover, by Hölder's inequality and Sobolev embedding, we get

$$\begin{aligned}\int_{B_\Omega} |u|^{m+1} dxdt &\le \int_0^T \|u\|_6^2 \|u\|^{m-1}_{\frac{3}{2}(m-1)} dt \le \int_0^T \|\nabla u\|_2^2 \|u\|^{m-1}_{\frac{3}{2}(m-1)} dt \\ &\le C\|u\|^{m-1}_{L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))} \int_0^T E(t)dt. \end{aligned}\tag{4.46}$$

Substituting (4.45) and (4.46) into (4.44) and using Young's inequality, we obtain that for any $\varepsilon > 0$,

$$\int_{B_\Omega} |g_1(u_t)u| dxdt \le \varepsilon \|u\|^{m-1}_{L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))} \int_0^T E(t)dt + C_\varepsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt. \tag{4.47}$$

Inserting (4.41) and (4.47) into (4.40), we get that for any $\varepsilon > 0$,

$$\begin{aligned} I_6 &= \frac{1}{2}\int_0^T \int_\Omega |g_1(u_t)u| dxdt \\ &\le \varepsilon \left(1 + \|u\|^{m-1}_{L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))}\right) \int_0^T E(t)dt + C_\varepsilon T|\Omega|\phi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right) \\ &\quad + C_\varepsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \quad \text{if } m > 5. \end{aligned}\tag{4.48}$$

We have finished estimating all terms I_i $(i = 1, ..., 6)$ from the right-hand side of equality (4.17).

Recalling that, if $m > 5$, we assume an extra regularity assumption on u that $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$. Then, by taking $\varepsilon > 0$ small enough, we can apply estimates (4.20), (4.25), (4.32), (4.33), (4.39), (4.43) and (4.48) to (4.17). It follows that

$$\frac{1-\xi}{2}\int_0^T E(t)dt \le \frac{cc_0}{c-2}(2\mathcal{E}(T) + D(T)) + C(\varepsilon, \alpha)(|\Omega| + |\Gamma|)T\Phi(D(T)), \tag{4.49}$$

where $T \ge T_0 \ge 1$. Note here the concave function $\Phi(s) := \phi_1(s) + \phi_2(s) + s$, and $D(T)$ is defined by (4.11). Also, the constant $0 < \xi < 1$ is defined by (4.24).

Noting that $\mathcal{E}(t)$ is non-increasing in time and $\mathcal{E}(t) \le E(t)$ for any $t \ge 0$, we have

$$T\mathcal{E}(T) \le \int_0^T \mathcal{E}(t)dt \le \int_0^T E(t)dt. \tag{4.50}$$

Because of (4.49) and (4.50), we see that

$$\Big(\frac{1-\xi}{2}T - \frac{2cc_0}{c-2}\Big)\mathcal{E}(T) \le \frac{cc_0}{c-2}D(T) + C(\varepsilon,\alpha)\Big(|\Omega| + |\Gamma|\Big)T\Phi(D(T)).$$

Since $T \ge T_0 \ge \frac{8cc_0}{(c-2)(1-\xi)}$, we obtain

$$\frac{1-\xi}{4}T\mathcal{E}(T) \le \frac{cc_0}{c-2}D(T) + C(\varepsilon,\alpha)\Big(|\Omega| + |\Gamma|\Big)T\Phi(D(T)). \tag{4.51}$$

Because $T \ge T_0 \ge 1$, then $\frac{1}{T} \le 1$, we infer from (4.51) that

$$\frac{1-\xi}{4}\mathcal{E}(T) \le \frac{cc_0}{c-2}D(T) + C(\varepsilon,\alpha)\Big(|\Omega| + |\Gamma|\Big)\Phi(D(T)). \tag{4.52}$$

Define the following constant $\tilde{C}$ independent of T:

$$\tilde{C} := \frac{4}{1-\xi}\left[\frac{cc_0}{c-2} + C(\varepsilon,\alpha)(|\Omega| + |\Gamma|)\right]. \tag{4.53}$$

Then we get from (4.52) that for any $T \ge T_0$,

$$\mathcal{E}(T) \le \tilde{C}\Phi(D(T)).$$

This completes the proof of Lemma 4.1.

4.3. *Completion of the proof of Theorem 2.14*

Now we use the stabilization estimate provided by Lemma 4.1 to prove Theorem 2.14. The strategy of the proof is adopted from paper [29] by Lasiecka and Tataru. The idea is to relate the stabilization estimate to an ODE, and the decay rate of the solution of the ODE determines the energy decay rate of the PDE.

Proof of Theorem 2.14. Let us fix a time $T \ge T_0$. From (4.14), we know that

$$\mathcal{E}(T) \le \tilde{C}\Phi(D(T)).$$

Note that $\mathcal{E}(T) + D(T) = \mathcal{E}(0)$. We define a concave function $\tilde{\Phi} : [0,\infty) \to [0,\infty)$ by

$$\tilde{\Phi}(s) := \tilde{C}\Phi(s) = \tilde{C}(\phi_1(s) + \phi_2(s) + s). \tag{4.54}$$

Notice that $\tilde{\Phi}$ is concave, increasing, continuous and satisfying $\tilde{\Phi}(0) = 0$. Then

$$\mathcal{E}(T) \leq \tilde{\Phi}(D(T)) = \tilde{\Phi}(\mathcal{E}(0) - \mathcal{E}(T)). \tag{4.55}$$

Hence, (4.55) yields

$$(I + \tilde{\Phi}^{-1})\mathcal{E}(T) \leq \mathcal{E}(0). \tag{4.56}$$

Here, $\tilde{\Phi}^{-1}$ is convex, increasing, continuous and vanishing at the origin.

Recall that $(u(t), w(t)) \in \mathcal{W}_1^\delta$ and $\mathcal{E}(t) \leq \mathcal{E}(0) \leq \Lambda(s^* - \delta)$ for all $t \geq 0$. Therefore, we can iterate (4.56) on $[mT, (m+1)T]$, $m = 0, 1, 2, \ldots$, to obtain

$$(I + \tilde{\Phi}^{-1})\mathcal{E}((m+1)T) \leq \mathcal{E}(mT), \quad m = 0, 1, 2, \ldots.$$

Following [29, Lemma 3.3], we have

$$\mathcal{E}(mT) \leq \sigma(m), \quad m = 0, 1, 2, 3\ldots. \tag{4.57}$$

Here $\sigma(t)$ is the solution of the ODE

$$\begin{cases} \sigma'(t) + [I - (I + \tilde{\Phi}^{-1})^{-1}]\sigma(t) = 0, \\ \sigma(0) = \mathcal{E}(0). \end{cases} \tag{4.58}$$

Since $I - (I + \tilde{\Phi}^{-1})^{-1} = (I + \tilde{\Phi})^{-1}$, we can reduce (4.58) to

$$\begin{cases} \sigma'(t) + (I + \tilde{\Phi})^{-1}\sigma(t) = 0, \\ \sigma(0) = \mathcal{E}(0), \end{cases} \tag{4.59}$$

where (4.59) has a unique solution on $[0, \infty)$. Noting that $\tilde{\Phi}$ is increasing and vanishing at the origin, then $(I + \tilde{\Phi})^{-1}$ is also increasing and vanishing at the origin. Rewrite (4.59) as $\sigma'(t) = -(I + \tilde{\Phi})^{-1}\sigma(t)$ to obtain $\sigma(t)$ is decreasing and approaching zero from above as $t \to \infty$.

For any $0 < T < t$, there is an $m \in \mathbb{N}$ such that $t = mT + \delta$, $0 \leq \delta < T$, hence $m = \frac{t}{T} - \frac{\delta}{T} > \frac{t}{T} - 1$. Since $\mathcal{E}(t)$ and $\sigma(t)$ are monotone decreasing, we can infer from (4.57) that for any $t > T$,

$$\mathcal{E}(t) = \mathcal{E}(mT + \delta) \leq \mathcal{E}(mT) \leq \sigma(m) \leq \sigma\left(\frac{t}{T} - 1\right). \tag{4.60}$$

(i) If g_1 and g_2 are linearly bounded near the origin, we see from (4.4) that ϕ_1 and ϕ_2 are linear, and hence $\tilde{\Phi}$ is also linear because of (4.54). Therefore $(I + \tilde{\Phi})^{-1}$ is a linear function. Then, (4.59) implies that for a constant $\gamma > 0$,

$$\begin{cases} \sigma'(t) + \gamma\sigma(t) = 0, \\ \sigma(0) = \mathcal{E}(0), \end{cases}$$

Then one has

$$\sigma(t) = \mathcal{E}(0)e^{-\gamma t}.$$

By using (4.60), we obtain that for any $t > T$,

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-\gamma(\frac{t}{T}-1)} = e^{\gamma}\mathcal{E}(0)e^{-\frac{\gamma}{T}t}. \tag{4.61}$$

Putting $a = \frac{\gamma}{T}$, we get (2.38).

(ii) We consider the case that at least one of g_1 and g_2 are either superlinear or sublinear near the origin. By (4.7) we can select $\phi_1(s) = C_1 s^{\nu_1}$ and $\phi_2(s) = C_2 s^{\nu_2}$, where $\nu_1, \nu_2 \in (0, 1)$ are given in (4.8).

Note that if $\lambda = (I + \tilde{\Phi})^{-1}(s)$ for $s \ge 0$, then $\lambda \ge 0$. Moreover, for any $0 \le \lambda \le 1$,

$$\begin{aligned} s &= (I + \tilde{\Phi})\lambda = \lambda + \tilde{C}\,(\phi_1(\lambda) + \phi_2(\lambda) + \lambda) \\ &\le C\,(\phi_1(\lambda) + \phi_2(\lambda) + \lambda) \le C\lambda^{\min\{\nu_1,\nu_2\}}. \end{aligned}$$

Then there exists $C_0 > 0$ such that $\lambda \ge C_0 s^j$ for any $0 \le \lambda \le 1$ where $j = \max\{\frac{1}{\nu_1}, \frac{1}{\nu_2}\} > 1$, namely,

$$(I + \tilde{\Phi})^{-1}(s) \ge C_0 s^j \ \text{ if } \ 0 \le (I + \tilde{\Phi})^{-1}(s) \le 1. \tag{4.62}$$

Noting that $(I + \tilde{\Phi})^{-1}(\sigma(t))$ is decreasing to zero as $t \to \infty$, then there exists $t_0 \ge 0$ such that $(I + \tilde{\Phi})^{-1}(\sigma(t)) \le 1$, whenever $t \ge t_0$. From (4.62), it follows that for $t \ge t_0$,

$$\sigma'(t) = -(I + \tilde{\Phi})^{-1}(\sigma(t)) \le -C_0\sigma^j(t).$$

Therefore, for all $t \ge t_0$, $\sigma(t) \le \tilde{\sigma}(t)$, where $\tilde{\sigma}(t)$ is the solution of

$$\begin{cases} \tilde{\sigma}'(t) + C_0\tilde{\sigma}^j(t) = 0, \\ \tilde{\sigma}(t_0) = \sigma(t_0), \end{cases}$$

from which we get for all $t \ge t_0$,

$$\tilde{\sigma}(t) = [C_0(j-1)(t - t_0) + \sigma^{1-j}(t_0)]^{-\frac{1}{j-1}},$$

which, along with (4.60), implies that for any $t \ge (t_0 + 1)T$,

$$\mathcal{E}(t) \le \sigma\left(\frac{t}{T} - 1\right) \le \tilde{\sigma}\left(\frac{t}{T} - 1\right) = \left[C_0(j-1)\left(\frac{t}{T} - 1 - t_0\right) + \sigma^{1-j}(t_0)\right]^{-\frac{1}{j-1}}.$$

Since $\sigma(t_0)$ depends on $\mathcal{E}(0)$, there exists a positive constant $C(\mathcal{E}(0))$ depending on $\mathcal{E}(0)$ such that for any $t \ge 0$,

$$\mathcal{E}(t) \le C(\mathcal{E}(0))(1 + t)^{-\frac{1}{j-1}}.$$

The proof of energy decay rates is completed. □

Data availability

No data was used for the research described in the article.

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