1. Mathematical Induction and Combinatorics

(1) Show that for each positive integer n, we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(2) Show that the cube of a positive integer can always be written as the difference of two squares.

(3) Establish a formula for $\displaystyle\sum_{k=2}^{n} \frac{1}{k^2-1}$ valid for each positive integer n.

(4) Establish a formula allowing one to obtain the sum of the first n positive even integers.

(5) Show that the formula $\displaystyle\sum_{j=1}^{n} (-1)^j j^2 = (-1)^n \sum_{j=1}^{n} j$ holds for each positive integer n.

(6) Show that $a+b$ is a factor of $a^{2n-1} + b^{2n-1}$ for each integer $n \geq 1$.

(7) Show that $a^2 + b^2$ is a factor of $a^{4n} - b^{4n}$ for each integer $n \geq 1$.

(8) Show that for each positive integer n,

$$a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

(9) Show that $\displaystyle\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$ for each positive integer n.

(10) Prove, using induction, that $(2n)! < 2^{2n}(n!)^2$ for each integer $n \geq 1$.

(11) Use induction in order to prove that $n^3 < n!$ for each integer $n \geq 6$.

(12) Let θ be a real number such that $\theta \geq -1$. Prove, using induction, that for each integer $n \geq 0$, we have $(1+\theta)^n \geq 1 + n\theta$.

(13) Let θ be a nonnegative real number. Show, using induction, that for each positive integer n, we have $(1+\theta)^n \geq 1 + n\theta + \frac{n(n-1)}{2}\theta^2$.

(14) Show that for each positive integer n, $\frac{1}{3}(n^3 + 2n)$ is an integer.

(15) Show that $\dfrac{10^n + 3 \cdot 4^{n+2} + 5}{9}$ is an integer for each positive integer n.

(16) Show that if n is a positive integer, then

$$\binom{n}{k} = \binom{n}{k+1} \iff n = 2k+1.$$

(17) Show that if n is a positive integer, then

(a) $\displaystyle\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = 2^{2n-1}$;

(b) $\displaystyle\binom{2n}{1} + \binom{2n}{3} + \cdots + \binom{2n}{2n-1} = 2^{2n-1}$.

(18) Prove that for each integer $n \geq 1$, we have

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

(19) Show that each integer $n > 7$ can be written as a sum containing only the numbers 3 and 5. For example, $8 = 3+5$, $9 = 3+3+3$, $10 = 5+5$.

(20) Assume that amongst n points, $n \geq 2$, in a given plane, no three points are on the same line. Show that the number of possible lines passing through these points is $n(n-1)/2$.

(21) Show that for each integer $n \geq 2$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

(22) Prove that for each positive integer k,

$$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2-1).$$

(23) We saw in problem 1 that, for each integer $n \geq 1$,

$$\begin{aligned} 1+2+\cdots+n &= \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}; \\ 1^2+2^2+\cdots+n^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}; \\ 1^3+2^3+\cdots+n^3 &= \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}. \end{aligned}$$

Hence, letting $S_k(n) = 1^k + 2^k + \cdots + n^k$ and in light of these three relations, it is normal to conjecture that, for each integer $k \geq 1$, $S_k(n)$ is a polynomial of degree $k+1$. In fact, in 1654, Blaise Pascal (1623–1662) established that indeed it was the case. His proof used induction and the expansion of the expression $(n+1)^{k+1} - 1$. Provide the details.

(24) Find a formula, valid for each integer $n \geq 2$, for

$$\prod_{i=2}^{n}\left(1 - \frac{1}{i}\right), \quad \text{and the same for} \quad \prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right).$$

(25) Show that, whatever the value of the integer $n \geq 1$, we always have

$$\sum_{i=1}^{n} \frac{i}{i^4 + i^2 + 1} < \frac{1}{2}.$$

(26) Show that if m, n and r are three positive integers such that

$$S := \frac{1}{m} + \frac{1}{n} + \frac{1}{r} < 1, \quad \text{then} \quad S \leq \frac{41}{42}.$$

(27) Given a positive integer n, let $s(n)$ be the sum of its digits (in basis 10). For each pair of positive integers k, ℓ smaller than 10, let $A_k(\ell)$ be the number of ℓ-digit positive integers n whose sum of digits is equal to k. In other words,

$$A_k(\ell) = \#\{n : 10^{\ell-1} \leq n < 10^{\ell},\ s(n) = k\}.$$

Show that

$$A_k(\ell) = \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1},$$

and conclude in particular that $A_k(\ell) = A_\ell(k)$.

(28) Using induction, prove the formulas due to Mariares (1913):

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \cdots + n^2 &= \binom{n+2}{3}, \quad \text{if } n \text{ is odd;} \\ 2^2 + 4^2 + 6^2 + \cdots + n^2 &= \binom{n+2}{3}, \quad \text{if } n \text{ is even.} \end{aligned}$$

(29) Let S be a set of 10 distinct integers chosen amongst the numbers $1, 2, \ldots, 99$. Show that S must contain two disjoint subsets for which the sum of their respective elements is the same.

(30) Given 51 arbitrary positive integers, show that one can always find two of them whose difference is 50.

(31) In order to acquire problem solving skills, a student decides to solve at least one problem per day and at most 11 per week and to do this for a whole year. Show that there exists a period of consecutive days during which he will solve exactly 20 problems.

(32) On a rectangular table of dimension 120 inches by 150 inches, we set 14 001 marbles. Show that, no matter how these are arranged, one can place a cylindrical glass with a diameter of 5 inches over at least 8 marbles.

(33) Choose n points on a circle and join them pairwise by secants. Taking for granted that no more than two secants can meet at the same point, in how many regions is the circle thus divided?

(34) Say we have three posts and n disks of different diameters placed on one of the posts, ordered by increasing diameters, the largest at the bottom of the post, the smaller at the top. The problem consists in transferring the tower of disks from the first post to the third post, using if need be the second post, but in such a way that, with each move, we do not place the moving disk on a smaller one. Establish the function of n which gives the minimum number of moves. (This problem is known as the "Tower of Hanoi Problem".)

(35) Let $\{F_n : n \in \mathbb{N}\}$ be the sequence of Fibonacci numbers defined by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that each positive integer can be written as the sum of distinct Fibonacci numbers.

(36) One easily checks that

$$\begin{aligned} 1 &= 1^2, \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2, \\ 3 &= -1^2 + 2^2, \\ 4 &= -1^2 - 2^2 + 3^2, \\ 5 &= 1^2 + 2^2, \\ 6 &= 1^2 - 2^2 + 3^2. \end{aligned}$$

Hence, we may be tempted to formulate a conjecture, namely that each positive integer n can be written as

$$n = e_1 1^2 + e_2 2^2 + e_3 3^2 + e_4 4^2 + \cdots + e_k k^2,$$

for a certain positive integer k (depending on n), where the $e_i \in \{-1, 1\}$. Prove this conjecture.

2. Divisibility

(37) The mathematician Duro Kurepa defined $!n = 0! + 1! + \cdots + (n-1)!$ for $n \geq 1$ and conjectured that $(!n, n!) = 2$ for all $n \geq 2$. This conjecture has been verified by Ivić and Mijajlović [20] for $n < 10^6$. Using computer software, write a program showing that this conjecture is true up to $n = 1000$.

(38) Consider the situation where the positive integer a is divided by the positive integer b using the euclidian division (see Theorem 7) yielding

$$(*) \qquad a = 652b + 8634.$$

By how much can we increase both a and b without changing the quotient $q = 652$?

(39) Consider the number $N = 111\ldots11$, here written in basis 2. Write N^2 in basis 2.

(40) Show that $39|7^{37} + 13^{37} + 19^{37}$.

(41) Show that, for each integer $n \geq 1$, the number $49^n - 2352n - 1$ is divisible by 2304.

(42) Given any integer $n \geq 1$, show that the number $n^4 + 2n^3 + 2n^2 + 2n + 1$ is never a perfect square.

(43) Let N be a two digit number. Let M be the number obtained from N by interchanging its two digits. Show that 9 divides $M - N$ and then find all the integers N such that $|M - N| = 18$.

(44) Is it true that 3 never divides $n^2 + 1$ for every positive integer n? Explain.

(45) Is it true that 5 never divides $n^2 + 2$ for every positive integer n? Explain. Is the result the same if one replaces the number 5 by the number 7?

(46) Given $s + 1$ integers $a_0, a_1, \ldots, a_s$ and a prime number p, show that p divides the integer

$$N(n) := a_0 + a_1 n + \cdots + a_{s-1} n^{s-1} + a_s n^s$$

if and only if p divides $N(r)$, for an integer r, $0 \leq r \leq p - 1$. Use this to find all integers n such that 7 divides $3n^2 + 6n + 5$.

(47) Compute the value of the expression

$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}.$$

(48) Show that, in any basis, the number 10101 is composite.

(49) Show that the product of four consecutive integers is necessarily divisible by 24.

(50) Show that the number

$$1^{47} + 2^{47} + 3^{47} + 4^{47} + 5^{47} + 6^{47}$$

is a multiple of 7.

(51) Show that the product of any five consecutive positive integers cannot be a perfect square.

(52) Show that $30|n^5 - n$ for each positive integer n.

(53) Show that $6|n(n + 1)(2n + 1)$ for each positive integer n.

(54) Given any integer $n \geq 0$, show that $64^{n+1} - 63n - 64$ is divisible by 3969. More generally, given $a \in \mathbb{N}$, show that for each integer $n \geq 0$, $(a+1)^{n+1} - an - (a+1)$ is divisible by a^2.
(55) Find all positive integers n such that $(n+1)|(n^2+1)$.
(56) Find all positive integers n such that $(n^2+2)|(n^6+206)$.
(57) Identify, if any exist, the positive integers n such that $(n^3+2)|(n^6+216)$.
(58) If a and b are positive integers such that $b|(a^2+1)$, do we necessarily have that $b|(a^4+1)$? Explain.
(59) Let n and k be positive integers.
(a) For $n \geq k$, show that

$$\frac{n}{(n,k)} \,\Big|\, \binom{n}{k}.$$

(b) For $n \geq k$, show that

$$\frac{n+1-k}{(n+1,k)} \,\Big|\, \binom{n}{k}.$$

(c) For $n \geq k-1 \geq 1$, show that

$$\frac{(n+1,k-1)}{n+2-k}\binom{n}{k-1} \quad \text{is an integer.}$$

(60) For each integer $n \geq 1$, let $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n), \qquad \text{for each integer } n \geq 1.$$

(61) Show that, for each positive integer n,

$$49|2^{3n+3} - 7n - 8.$$

(62) Find all positive integers a for which $a^{10}+1$ is divisible by 10.
(63) Is it true that $3|2^{2n}-1$ for each positive integer n? Explain.
(64) Show that if an integer is of the form $6k+5$, then it is necessarily of the form $3k-1$, while the reverse is false.
(65) Can an integer $n > 1$ be of the form $8k+7$ and also of the form $6\ell+5$? Explain.
(66) Let $M_1 = 2+1$, $M_2 = 2\cdot 3+1$, $M_3 = 2\cdot 3\cdot 5+1$, $M_4 = 2\cdot 3\cdot 5\cdot 7+1$, $M_5 = 2\cdot 3\cdot 5\cdot 7\cdot 11+1, \ldots$. Prove none of the numbers M_k is a perfect square.
(67) Verify that if an integer is a square and a cube, then it must be of the form $7k$ or $7k+1$.
(68) If x and y are odd integers, prove that x^2+y^2 cannot be a perfect square.
(69) Show that, for each positive integer n, we have $n^2|(n+1)^n - 1$.
(70) Let $k, n \in \mathbb{N}$, $n \geq 2$. Show that $(n-1)^2|(n^k-1)$ if and only if $(n-1)|k$. More generally, show the following result: Let $a \in \mathbb{Z}$ and $k, n \in \mathbb{N}$ with $n \neq a$; then $(n-a)^2|(n^k-a^k)$ if and only if $(n-a)|ka^{k-1}$.
(71) Let a, b be integers and let n be a positive integer.
(a) If $a - b \neq 0$, show that

$$\left(\frac{a^n-b^n}{a-b}, a-b\right) = \left(n(a,b)^{n-1}, a-b\right).$$

(b) If $a+b\neq 0$ and if n is odd, show that

$$\left(\frac{a^n+b^n}{a+b}, a+b\right) = \left(n(a,b)^{n-1}, a+b\right).$$

(c) Show that if a and b are relatively prime with $a+b\neq 0$ and if $p>2$ is a prime number, then

$$\left(\frac{a^p+b^p}{a+b}, a+b\right) = \begin{cases} 1 & \text{if } p\nmid(a+b), \\ p & \text{if } p|(a+b). \end{cases}$$

(72) Let k and n be positive integers. Show that the only solutions (k,n) of the equation $(n-1)! = n^k - 1$ are $(1,2)$, $(1,3)$ and $(2,5)$.

(73) According to Euclid's algorithm, assuming that $b \geq a$ are positive integers, we have

$$\begin{aligned} b &= aq_1 + r_1, & 0 < r_1 < a, \\ a &= r_1q_2 + r_2, & 0 < r_2 < r_1, \\ r_1 &= r_2q_3 + r_3, & 0 < r_3 < r_2, \\ &\vdots & \\ r_{j-2} &= r_{j-1}q_j + r_j, & 0 < r_j < r_{j-1}, \\ r_{j-1} &= r_jq_{j+1}, & \end{aligned}$$

where $r_j = (a,b)$.

(a) Show that $b > 2r_1$, $a > 2r_2$ and for $k \geq 1$, $r_k > 2r_{k+2}$.

(b) Deduce that $b > 2^{j/2}$ and therefore that the maximum number of steps in Euclid's algorithm is $[2(\log b/\log 2)]$.

(74) Show that there exist infinitely many positive integers n such that $n|2^n+1$.

(75) Let a be an integer ≥ 2. Show that for positive integers m and n we have

$$a^n - 1 | a^m - 1 \iff n|m.$$

(76) Let N_n be an integer formed of n consecutive "1"s. For example, $N_3 = 111$, $N_7 = 1\,111\,111$. Show that $N_n|N_m \iff n|m$.

(77) Prove that no member of the sequence $11, 111, 1\,111, 11\,111, \ldots$ is a perfect square.

(78) What is the smallest positive integer divisible both by 2 and 3 which is both a perfect square and a sixth power? More generally, what is the smallest positive integer n divisible by both 2 and 3 which is both an n–th power and an m–th power, where $n, m \geq 2$?

(79) Three of the four integers, found between 100 and 1000, with the property of being equal to the sum of the cubes of their digits are 153, 370 and 407. What is the fourth of these integers?

(80) How many positive integers $n \leq 1000$ are not divisible by 2, nor by 3, nor by 5?

(81) Prove the following result obtained in the seventeenth century by Pierre de Fermat (1601–1665): *"Each odd prime number p can be written as the difference of two perfect squares."*

(82) Prove that the representation mentioned in problem 81 is unique.

(83) Is the result of Fermat stated in problem 81 still true if p is simply an odd positive integer?

(84) Let $n = 999\,980\,317$. Observing that $n = 10^9 - 3^9$ and factoring this last expression, conclude that $7|n$.

(85) Show that if an odd integer can be written as the sum of two squares, then it is of the form $4n+1$.

(86) Let $a,b,c \in \mathbb{Z}$ be such that $abc \neq 0$ and $(a,b,c) = 1$ and such that $a^2+b^2=c^2$. Prove that at least one of the integers a and b is even.

(87) For which integer values of k is the number 10^k-1 the cube of an integer?

(88) Show that if the positive integer a divides both $42n+37$ and $7n+4$ for a certain integer n, then $a=1$ or $a=13$.

(89) If a and b are two positive integers and if $\frac{1}{a}+\frac{1}{b}$ is an integer, prove that $a=b$. Moreover, show that a is then necessarily equal to 1 or 2.

(90) Let $a,b \in \mathbb{N}$ such that $(a,b)=4$. Find all possible values of (a^2,b^3).

(91) Let $a,b \in \mathbb{N}$ and $d=(a,b)$. Find the value of $(3a+5b, 5a+8b)$ in terms of d and more generally that of $(ma+nb, ra+sb)$ knowing that $ms-nr=1$, where $m,n,r,s \in \mathbb{N}$.

(92) Let $m,n \in \mathbb{N}$. If $d|mn$ where $(m,n)=1$, show that d can be written as $d=rs$ where $r|m$, $s|n$ and $(r,s)=1$.

(93) Let a,b,d be nonzero integers, d odd, such that $d|(a+b)$ and $d|(a-b)$. Show that $d|(a,b)$.

(94) Given eight positive composite integers ≤ 360, show that at least two of them have a common factor larger than 1.

(95) If a and b are positive integers such that $(a,b)=1$ and ab is a perfect square, show that a and b are perfect squares.

(96) Can $n(n+1)$ be a perfect square for a certain positive integer n? Explain.

(97) What are the possible values of the expression $(n, n+14)$ as n runs through the set of positive integers?

(98) Let $n>1$ an integer. Which of the following statements are true:

$$3|(n^3-n), \qquad 3|n(n+1), \qquad 8|(2n+1)^2-1, \qquad 6|n(n+1)(n+2).$$

(99) Is it true that if n is an even integer, then $24|n(n+1)(n+2)$? Explain.

(100) Let n be an integer such that $(n,2)=(n,3)=1$. Show that $24|n^2+47$.

(101) Let $d=(a,b)$, where a and b are positive integers. Show that there are exactly d numbers amongst the integers $a, 2a, 3a, \ldots, ba$ which are divisible by b.

(102) Let a,b be integers such that $(a,b)=d$, and let x_0,y_0 be integers such that $ax_0+by_0=d$. Show that:
(a) $(x_0,y_0)=1$;
(b) x_0 and y_0 are not unique.

(103) Let a,m and n be positive integers. If $(m,n)=1$, show that $(a,mn)=(a,m)(a,n)$.

(104) For all $n \in \mathbb{N}$, show that $(n^2+3n+2, 6n^3+15n^2+3n-7)=1$.

(105) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, show that
(a) $(a+b,a-b)=1$ or 2; (b) $(2a+b,a+2b)=1$ or 3;
(c) $(a^2+b^2,a+b)=1$ or 2; (d) $(a+b,a^2-3ab+b^2)=1$ or 5.

(106) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, find the possible values of
(a) (a^3+b^3,a^3-b^3); (b) (a^2-b^2,a^3-b^3).

(107) Let a, b and c be integers. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.
(a) If $(a,b)=(a,c)$, then $[a,b]=[a,c]$.

(b) If $(a,b)=(a,c)$, then $(a^2,b^2)=(a^2,c^2)$.
(c) If $(a,b)=(a,c)$, then $(a,b)=(a,b,c)$.

(108) Let $a,b\in\mathbb{Z}$ and let $m,n\in\mathbb{N}$. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.
(a) If $a^n|b^n$, then $a|b$.
(b) If $a^m|b^n$, $m>n$, then $a|b$.
(c) If $a^m|b^n$, $m<n$, then $a|b$.

(109) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,b)=1$ and $c|a$, then $(c,b)=1$.

(110) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,bc)=1$, then $(a,b)=(a,c)=1$.

(111) Let $a,b\in\mathbb{Z}$. Show that $(a,b)=(a+b,[a,b])$. Using this result, find two positive integers whose sum is 186 and whose LCM is 1440.

(112) Let $a,b,c\in\mathbb{Z}$.
(a) Show that $(a,bc)=(a,(a,b)c)$.
(b) Show that $(a,bc)=(a,(a,b)(a,c))$.

(113) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,c)=1$, then $(ab,c)=(b,c)$.

(114) Let a, b, m and n be integers. If $(m,n)=1$, show that $(ma+nb,mn)=(a,n)(b,m)$. Show that this result generalizes the result of problem 103.

(115) Is it possible that $\binom{n}{r}$ is relatively prime with $\binom{n}{s}$, for certain positive integers r,s,n satisfying $0<r<s\le n/2$? Explain.

(116) Find two positive integers for which the difference between their LCM and their GCD is equal to 143.

(117) Let a,b,c be positive integers. Show that $(a,b,c)=((a,b),c)$ and $[a,b,c]=[[a,b],c]$. Generalize this result. Use this result to compute $(132,102,36)$ and find those integers x,y,z for which $132x+102y+36z=(132,102,36)$.

(118) Let n be a positive integer. Evaluate $(n,n+1,n+2)$ and $[n,n+1,n+2]$.

(119) Let a,b,c be positive integers. If $(a,b)=(b,c)=(a,c)=1$, show that $(a,b,c)[a,b,c]=abc$.

(120) Is it true that if a and b are positive integers such that $(a,b)=1$, then $(a^2,ab,b^2)=1$? Explain.

(121) Is it true that if a, b and c are positive integers, then $[a^2,ab,b^2]=[a^2,b^2]$? Explain.

(122) Is it true that if a, b and c are positive integers, then $(a,b,c)=\big((a,b),(a,c)\big)$? Explain.

(123) Is it true that $[a,b,c]\cdot(a,b,c)=|abc|$, $\forall\ a,b,c\in\mathbb{Z}\setminus\{0\}$? Explain.

(124) Let a, b, d, m and n be positive integers such that $a|d^m-1$, $b|d^n-1$ and $(a,b)=1$. Show that $ab|d^{[m,n]}-1$.

(125) Show that if a is an integer >1, then, for each pair of positive integers m and n,

$$(a^m-1,a^n-1)=a^{(m,n)}-1.$$

What do we obtain for (a^m+1,a^n+1), for (a^m+1,a^n-1)? More generally, given $a>1$ and $b>1$, what are the values of

$$(a^m-b^m,a^n-b^n),\quad (a^m+b^m,a^n+b^n)\ \text{ and }\ (a^m+b^m,a^n-b^n)?$$

(126) Show that there exist infinitely many pairs of integers $\{x,y\}$ satisfying $x+y=40$ and $(x,y)=5$.

(127) Find all pairs of positive integers $\{a, b\}$ such that $(a, b) = 15$ and $[a, b] = 90$. More generally, if d and m are positive integers, show that there exists a pair of positive integers $\{a, b\}$ for which $(a, b) = d$ and $[a, b] = m$ if and only if $d|m$. Moreover, in this situation, show that the number of such pairs is 2^r, where r is the number of distinct prime factors of m/d.

(128) Prove that one cannot find integers m and n such that $m + n = 101$ and $(m, n) = 3$.

(129) Let $a, m, n \in \mathbb{N}$ with $m \neq n$.
(a) Show that $(a^{2^n} + 1)|(a^{2^m} - 1)$ if $m > n$.
(b) Show that $(a^{2^n} + 1, a^{2^m} + 1) = \begin{cases} 1 & \text{if } a \text{ is even,} \\ 2 & \text{if } a \text{ is odd.} \end{cases}$

(130) Let n be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \ldots, \binom{2n}{2n-1}.$$

(131) Given $n + 1$ distinct positive integers $a_1, a_2, \ldots, a_{n+1}$ such that $a_i \leq 2n$ for $i = 1, 2, \ldots, n + 1$, show that there exists at least one pair $\{a_j, a_k\}$ with $j \neq k$ such that $a_j | a_k$.

(132) Let $n > 2$. Consider the three n-tuples $(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)})$, $i = 1, 2, 3$, where each $a_j^{(i)} \in \{+1, -1\}$ and assume that these three n-tuples satisfy $\sum_{j=1}^{n} a_j^{(i)} a_j^{(k)} = 0$ for each pair $\{i, k\}$ such that $1 \leq i < k \leq 3$. Show that $4|n$.

(133) Let A be the set of natural numbers which, in their decimal representation, do not have "7" amongst their digits. Prove that
$$\sum_{n \in A} \frac{1}{n} < +\infty.$$

(134) Let $u_1, u_2, \ldots$ be a strictly increasing sequence of positive integers. Denoting by $[a, b]$ the lowest common multiple of a and b, show that the series
$$\sum_{n=1}^{\infty} \frac{1}{[u_n, u_{n+1}]} \qquad \text{converges.}$$

3. Prime Numbers

(135) Using computer software, write a program
(a) to generate all Mersenne primes up to $2^{525} - 1$;
(b) to determine the smallest prime number larger than $10^{100} + 1$.

(136) Write a program that generates prime numbers up to a given number N. One can, of course, use Eratosthenes' sieve.

(137) Use a computer to find four consecutive integers having the same number of prime factors (allowing repetitions).

(138) (a) By reversing the digits of the prime number 1009, we obtain the number 9001, which is also prime. Write a program to find the prime numbers in [1,10000] verifying this property.
(b) By reversing the digits of the prime number 163, we obtain the number 361, which is a perfect square. Using computer software, write a program to find all prime numbers in $[1, 10000]$ with this property.

(139) Using a computer, find all prime numbers $p \leq 10\,000$ with the property that p, $p+2$ and $p+6$ are all primes.

(140) Let p_k be the k-th prime number. Show that $p_k < 2^k$ if $k \geq 2$.

(141) If a prime number $p_k > 5$ is equally isolated from the prime numbers appearing before and after it, that is $p_k - p_{k-1} = p_{k+1} - p_k = d$, say, show that d is a multiple of 6. Then, for each of the cases $d = 6$, 12 and 18, find, by using a computer, the smallest prime number p_k with this property.

(142) Prove that none of the numbers

$$12321,\ 1234321,\ 123454321,\ 12345654321,\ 1234567654321,$$

$$123456787654321,\ 12345678987654321$$

is prime.

(143) For each integer $k \geq 1$, let n_k be the k-th composite number, so that for instance $n_1 = 4$ and $n_{10} = 18$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(144) For each integer $k \geq 1$, let n_k be the k-th number of the form p^α, where p is prime, α a positive integer, so that for instance $n_1 = 2$ and $n_{10} = 16$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(145) Find all positive integers $n < 100$ such that $2^n + n^2$ is prime. To which class of congruence modulo 6 do these numbers n belong?

(146) Show that if the integer $n \geq 4$ is not an odd multiple of 9, then the corresponding number $a_n := 4^n + 2^n + 1$ is necessarily composite. Then, use a computer in order to find all positive integers $n < 1000$ for which a_n is prime.

(147) Consider the sequence (a_n) defined by $a_1 = a_2 = 1$ and, for $n \geq 3$, by $a_n = n! - (n-1)! + \cdots + (-1)^n 2! + (-1)^{n+1} 1!$. Use a computer in order to find the smallest number n such that a_n is a composite number.

(148) The mathematicians Minác and Willans have obtained a formula for the n-th prime number p_n which is more of a theoretical interest than of a

practical interest:

$$p_n = 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{1 + \sum_{j=2}^{m} \left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j} \right] \right]} \right]^{1/n} \right],$$

where as usual $[x]$ stands for the largest integer $\leq x$. Prove this formula.

(149) Develop an idea used by Paul Erdős (1913–1996) to show that, for each integer $n \geq 1$,

$$\prod_{p \leq n} p \leq 4^n.$$

His idea was to write

$$\prod_{p \leq n} p = \prod_{p \leq \frac{n+1}{2}} p \cdot \prod_{\frac{n+1}{2} < p \leq n} p$$

and to use the fact that each prime number $p > (n+1)/2$ appears in the factorization of the binomial coefficient $\binom{n}{(n+1)/2}$. Provide the details.

(150) Show that if four positive integers a, b, c, d are such that $ab = cd$, then the number $a^2 + b^2 + c^2 + d^2$ is necessarily composite.

(151) Show that, for each integer $n \geq 1$, the number $4n^3 + 6n^2 + 4n + 1$ is composite.

(152) Show that if p and q are two consecutive odd prime numbers, then $p + q$ is the product of at least three prime numbers (not necessarily distinct).

(153) Does there exist a positive integer n such that $n/2$ is a perfect square, $n/3$ a cube and $n/5$ a fifth power?

(154) Given any integer $n \geq 2$, show that $n^{42} - 27$ is never a prime number.

(155) Let $\theta(x) := \sum_{p \leq x} \log p$. Prove that Bertrand's Postulate follows from the fact that

$$c_1 x < \theta(x) < c_2 x,$$

where $c_1 = 0.73$ and $c_2 = 1.12$.

(156) Use Bertrand's Postulate to show that, for each integer $n \geq 4$,

$$p_{n+1}^2 < p_1 p_2 \cdots p_n,$$

where p_n stands for the n-th prime number.

(157) Certain integers $n \geq 3$ can be written in the form $n = p + m^2$, with p prime and $m \in \mathbb{N}$. This is the case for example for the numbers 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21. Let q^r be a prime power, where r is a positive even integer such that $2q^{r/2} - 1$ is composite. Show that q^r cannot be written as $q^r = p + m^2$, with p prime and $m \in \mathbb{N}$.

(158) Show that if p and $8p - 1$ are primes, then $8p + 1$ is composite.

(159) Show that all positive integers of the form $3k + 2$ have a prime factor of the same form, that all positive integers of the form $4k + 3$ have a prime factor of the same form, and finally that all positive integers of the form $6k + 5$ have a prime factor of the same form.

(160) A positive integer n has a *Cantor expansion* if it can be written as

$$n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1!,$$

where the a_j's are integers satisfying $0 \leq a_j \leq j$.

(a) Find the Cantor expansion of 23 and of 57.

(b) Show that all positive integers n have a Cantor expansion and moreover that this expansion is unique.

(161) If $p > 1$ and $d > 0$ are integers, show that p and $p+d$ are both primes if and only if

$$(p-1)!\left(\frac{1}{p}+\frac{(-1)^d d!}{p+d}\right)+\frac{1}{p}+\frac{1}{p+d}$$

is an integer.

(162) Find all prime numbers p such that $p+2$ and p^2+2p-8 are primes.

(163) Is it true that if p and p^2+8 are primes, then p^3+4 is prime? Explain.

(164) Let $n \geq 2$. Show that the integers n and $n+2$ form a pair of twin primes if and only if

$$4\left((n-1)!+1\right)+n \equiv 0 \pmod{n(n+2)}.$$

(165) Identify each prime number p such that $2^p + p^2$ is also prime.

(166) For which prime number(s) p is $17p+1$ a perfect square?

(167) Given two integers a and b such that $(a,b) = p$, where p is prime, find all possible values of:

(a) (a^2, b); (b) (a^2, b^2); (c) (a^3, b); (d) (a^3, b^2).

(168) Given two integers a and b such that $(a, p^2) = p$ and $(b, p^4) = p^2$, where p is prime, find all possible values of:

(a) (ab, p^5); (b) $(a+b, p^4)$; (c) $(a-b, p^5)$; (d) $(pa-b, p^5)$.

(169) Given two integers a and b such that $(a, p^2) = p$ and $(b, p^3) = p^2$, where p is a prime number, evaluate the expressions (a^2b^2, p^4) and (a^2+b^2, p^4).

(170) Let p be a prime number and a, b, c be positive integers. For each of the following statements, say if is true or false. If it is true, give a proof; if it is false, provide a counter-example.

(a) If $p|a$ and $p|(a^2+b^2)$, then $p|b$.
(b) If $p|a^n$, $n \geq 1$, then $p|a$.
(c) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2-c^2)$.
(d) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2+c^2)$.

(171) Let a, b and c be positive integers. Show that $abc = (a,b,c)[ab,bc,ac] = (ab,bc,ac)[a,b,c]$.

(172) Let a, b and c be positive integers and assume that $abc = (a,b,c)[a,b,c]$. Show that this necessarily implies that $(a,b) = (b,c) = (a,c) = 1$.

(173) Let a, b and c be positive integers. Show that $(a,b,c) = \dfrac{(a,b)(b,c)(a,c)}{(ab,bc,ac)}$ and that $[a,b,c] = \dfrac{abc\,(a,b,c)}{(a,b)(b,c)(a,c)}$.

(174) Let a, b and c be positive integers. Show that

$$\frac{[a,b,c]^2}{[a,b][b,c][c,a]} = \frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}.$$

(175) Find three positive integers a, b, c such that

$$[a,b,c]\cdot(a,b,c) = \sqrt{abc}.$$

(176) Let $\#n = [1,2,3,\ldots,n]$ be the lowest common multiple of the numbers $1,2,\ldots,n$. Show that

$$\prod_{p\leq n} p \leq \#n = \prod_{p\leq n} p^{[\log n/\log p]}.$$

(177) Let p be a prime number and r a positive integer. What are the possible values of $(p, p+r)$ and of $[p, p+r]$?

(178) Let $p > 2$ be a prime number such that $p|8a-b$ and $p|8c-d$, where $a,b,c,d \in \mathbb{Z}$. Show that $p|(ad-bc)$.

(179) Show that, if $\{p, p+2\}$ is a pair of twin primes with $p > 3$, then 12 divides the sum of these two numbers.

(180) Let n be a positive integer. Show that if n is a composite integer, then $n|(n-1)!$ except when $n = 4$.

(181) For which positive integers n is it true that

$$\sum_{j=1}^{n} j \;\Big|\; \prod_{j=1}^{n} j?$$

(182) Let $\pi = 3.141592\ldots$ be Archimede's constant, and for each positive real number x, let $\pi_2(x)$ be the function that counts the number of pairs of twin primes $\{p, p+2\}$ such that $p \leq x$. Show that

$$\pi_2(x) = 2 + \sum_{7\leq n\leq x} \sin\left(\frac{\pi}{2}(n+2)\left[\frac{n!}{n+2}\right]\right)\cdot\sin\left(\frac{\pi}{2}n\left[\frac{(n-2)!}{n}\right]\right),$$

where $[y]$ stands for the largest integer $\leq y$.

(183) Given an integer $n \geq 2$, show, without using Bertrand's Postulate, that there exists a prime number p such that $n < p < n!$.

(184) In 1556, Niccòlo Tartaglia (1500–1557) claimed that the sums

$$1+2+4,\ 1+2+4+8,\ 1+2+4+8+16,\ \ldots$$

stood successively for a prime number and a composite number. Was he right?

(185) Show that if $a^n - 1$ is prime for certain integers $a > 1$ and $n > 1$, then $a = 2$ and n is prime.

REMARK: *The integers of the form $2^p - 1$, where p is prime, are called Mersenne numbers. We denote them by M_p in memory of Marin Mersenne* (1588–1648), *who had stated that M_p is prime for*

$$p = 2,3,5,7,13,17,19,31,67,127,257$$

and composite for all the other primes $p < 257$. This assertion of Mersenne can be found in the preface of his book Cogita Physico-mathematica, *published in Paris in 1644. Since then, we have found a few errors in the computations of Mersenne: indeed M_p is not prime for $p = 67$ and $p = 257$, while M_p is prime for $p = 61$, $p = 89$ and $p = 109$. One can find in the appendix C of the book of J.M. De Koninck and A. Mercier* [**8**] *the list of Mersenne primes M_p corresponding to the prime numbers p satisfying* $2 \leq p \leq 44\,497$. *Note on the other hand that it has recently been discovered that $2^{32\,582\,657} - 1$ is prime (in September 2006), which brings to 44 the total number of known Mersenne primes. It is also known that the primes*

M_p are closely related to the PERFECT NUMBERS, *in the sense that, as was shown by Leonhard Euler (1707–1783), n is an even perfect number if and only if $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.*

(186) Show that if there exists a positive integer n and an integer $a \geq 2$ such that $a^n + 1$ is prime, then a is even and $n = 2^r$ for a certain positive integer r.

REMARK: *The prime numbers of the form $2^{2^k} + 1$, $k = 0, 1, 2, \ldots$, are called "Fermat primes". The reason is that Pierre de Fermat claimed in 1640 (although saying he could not prove it) that all the numbers of the form $2^{2^k} + 1$ are prime. One hundred years later, Euler proved that*

$$2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417.$$

As of today, we still do not know if, besides the cases $k = 0, 1, 2, 3, 4$, primes of the form $2^{2^k} + 1$ exist. Nevertheless, it is known that $2^{2^k} + 1$ is composite for $5 \leq k \leq 32$; see H.C. Williams **[41]** *and the site* www.prothsearch.net/fermat.html.

(187) Show that the equation $(2^x - 1)(2^y - 1) = 2^{2^z} + 1$ is impossible for positive integers x, y and z. (This implies in particular that a Fermat number, that is a number of the form $2^{2^k} + 1$, cannot be the product of two Mersenne numbers.)

(188) Prove by induction that, for each integer $n \geq 1$,

$$F_0 F_1 F_2 \cdots F_{n-1} = F_n - 2,$$

where $F_i = 2^{2^i} + 1$, $i = 0, 1, 2, \ldots$.

(189) Use the result of problem 188 in order to prove that if m and n are distinct positive integers, then $(F_m, F_n) = 1$.

(190) A positive integer n is said to be *pseudoprime in basis* $a \geq 2$ if it is composite and if $a^{n-1} \equiv 1 \pmod{n}$. Find the smallest number which is pseudoprime in each of the bases 2, 3, 5 and 7.

(191) Use Problem 189 to prove that there exist infinitely many primes.

(192) Consider the numbers $f_n = 2^{3^n} + 1$, $n = 1, 2, \ldots$, and show they are all composite and in particular that, for each positive integer n,
(a) $3^{n+1} | f_n$; (b) $p | f_n \Rightarrow p | f_{n+1}$.

(193) Show that there exist infinitely many prime numbers p such that the numbers $p - 2$ and $p + 2$ are both composite.

(194) Show that 641 divides $F_5 = 2^{2^5} + 1$ without doing the explicit division.

(195) Use an induction argument in order to prove that each Fermat number $F_n = 2^{2^n} + 1$, where $n \geq 2$, ends with the digit 7.

(196) Let n be a positive integer and consider the set $E = \{1, 2, \ldots, n\}$. Let 2^k be the largest power of 2 which belongs to E. Show that for all $m \in E \setminus \{2^k\}$, we have $2^k \nmid m$. Using this result, show that $\sum_{j=1}^{n} 1/j$ is not an integer if $n > 1$.

(197) Show that, for each positive integer n, one can find a prime number $p < 50$ such that $p | (2^{5n} - 1)$.

(198) Show that the integers defined by the sequence of numbers

$$M_k = p_1 p_2 \cdots p_k + 1 \qquad (k = 1, 2, \ldots),$$

where p_j stands for the j–th prime number, are prime numbers for $1 \leq k \leq 5$ and composite numbers for $k = 6, 7$. What about M_8, M_9 and M_{10}?

(199) Use the proof of Euclid's Theorem on the infinitude of primes to show that, if we denote by p_r the r-th prime number, then $p_r \leq 2^{2^{r-1}}$ for each $r \in \mathbb{N}$.

(200) In Problem 199, we obtained an upper bound for p_r, the r-th prime number, namely $p_r \leq 2^{2^{r-1}}$. Use this inequality to obtain a lower bound for $\pi(x)$, the number of prime numbers $\leq x$. More precisely, show that, for $x \geq 3$, $\pi(x) \geq \log \log x$.

(201) Show that there exist infinitely many prime numbers of the form $4n + 3$.

(202) Show that there exist infinitely many prime numbers of the form $6n + 5$.

(203) Let $f \colon \mathbb{N} \to \mathbb{R}$ be the function defined by

$$f(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0,$$

where $a_r \neq 0$ and where each a_i, $0 \leq i \leq r$, is an integer. Show that, by an appropriate choice of a_i, $0 \leq i \leq r$, the set $\{f(n) : n \in \mathbb{N}\}$ contains at least r prime numbers.

(204) Consider the positive integers which can be written as an alternating sequence of 0's and 1's. The number 101 010 101 is such a number and observe that $101\,010\,101 = 41 \cdot 271 \cdot 9091$. Besides 101, do there exist other prime numbers of this form?

(205) Find all prime numbers of the form $2^{2^n} + 5$, where $n \in \mathbb{N}$. Would the question be more difficult if one replaces the number 5 by another number of the form $3k + 2$? Explain.

(206) The largest gaps between two consecutive prime numbers $p_r < p_{r+1} < 100$ occur successively when

$$\begin{aligned} p_{r+1} - p_r &= 5 - 3 = 2, \\ p_{r+1} - p_r &= 11 - 7 = 4, \\ p_{r+1} - p_r &= 29 - 23 = 6, \\ p_{r+1} - p_r &= 97 - 89 = 8. \end{aligned}$$

Is it true that these constantly increasing gaps always occur by jumps of length 2? In other words, does the first gap of length $2k$ always occur before the first gap of length $2k + 2$?

(207) Show that $\displaystyle\sum_{\alpha=2}^{\infty} \sum_{p} \frac{1}{p^\alpha} < 1$, where the inner sum runs over all the prime numbers p.

(208) Let

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \cdots,$$

be a series which is in fact a finite sum for each real number $x \geq 1$ since $\pi(x^{1/n}) = 0$ as soon as $n > \log x / \log 2$. Show that

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}).$$

REMARK: It is possible to show that $f(x)$ is a better approximation of $\pi(x)$ than $Li(x) := \int_2^x \frac{dt}{\log t}$ (see H. Riesel [**31**]).

(209) Let $n \geq 2$ be an integer. Show that the interval $[n, 2n]$ contains at least one perfect square.

(210) If n is a positive integer such that $3n^2 - 3n + 1$ is composite, show that n^3 cannot be written as $n^3 = p + m^3$, with p prime and m a positive integer.

(211) It is conjectured that there exist infinitely many prime numbers p of the form $p = n^2 + 1$. Identify the primes $p < 10\,000$ of this particular form. Why is the last digit of such a prime number p always 1 or 7? Is there any reasonable explanation for the fact that the digit 7 appears essentially twice as often?

(212) Show that, for each integer $n \geq 2$,

$$(n!)^{1/n} \leq \prod_{p \leq n} p^{\frac{1}{p-1}}.$$

(213) For each integer $N \geq 1$, let $S_N = \{n^2 + 2 : 6 \leq n \leq 6N\}$. Show that no more than $\frac{1}{6}$ of the elements of S_N are primes.

(214) Let p be a prime number and consider the integer $N = 2 \cdot 3 \cdot 5 \cdots p$. Show that the $(p-1)$ consecutive integers

$$N + 2, N + 3, N + 4, \ldots, N + p$$

are composite.

(215) Let $n > 1$ be an integer with at least 3 digits. Show that
- (a) $2|n$ if and only if the last digit of n is divisible by 2;
- (b) $2^2|n$ if and only if the number formed with the last two digits of n is divisible by 4;
- (c) $2^3|n$ if and only if the number formed with the last three digits of n is divisible by 8.

Can one generalize?

(216) For each integer $n \geq 2$, let

$$P(n) = \prod_{\substack{p|n \\ p > \log n}} \left(1 - \frac{1}{p}\right).$$

Show that $\lim_{n\to\infty} P(n) = 1$.

(217) Prove that there exists an interval of the form $[n^2, (n+1)^2]$ containing at least 1000 prime numbers.

(218) Use the Prime Number Theorem (see Theorem 17) in order to prove that the set of numbers of the form p/q (where p and q are primes) is dense in the set of positive real numbers.

(219) Show that the sum of the reciprocals of a finite number of distinct prime numbers cannot be an integer.

(220) Use the fact that there exists a positive constant c such that if $x \geq 100$,

$$(1) \qquad \sum_{p \leq x} \frac{1}{p} = \log\log x + c + R(x) \quad \text{with } |R(x)| < \frac{1}{\log x}$$

and moreover that, for $x \geq 2$,

$$\text{(2)} \qquad \pi(x) := \sum_{p \leq x} 1 < \frac{3}{2} \frac{x}{\log x}$$

in order to prove that if $P(n)$ stands for the largest prime factor of n, then

$$\text{(3)} \qquad \frac{1}{x} \#\{n \leq x : P(n) > \sqrt{x}\} = \log 2 + T(x) \quad \text{with } |T(x)| < \frac{9}{2} \frac{1}{\log x}.$$

Use this result to show that more than $\frac{2}{3}$ of the integers have their largest prime factor larger than their square root, or in other words that the density of the set of integers n such that $P(n) > \sqrt{n}$ is larger than $\frac{2}{3}$.

(221) Prove the following formula (due to Adrien-Marie Legendre (1752–1833)):

$$\pi(x) = \pi(\sqrt{x}) + \sum_{n | p_1 \cdots p_r} \mu(n) \left[\frac{x}{n}\right] - 1,$$

where $r = \pi(\sqrt{x})$.

(222) Consider the following two conjectures:

A. *(Goldbach Conjecture) Each even integer ≥ 4 can be written as the sum of two primes.*

B. *Each integer > 5 can be written as the sum of three prime numbers.*

Show that these two conjectures are equivalent.

(223) Show that $\pi(m)$, the number of prime numbers not exceeding the positive integer m, satisfies the relation

$$\pi(m) = \sum_{j=2}^{m} \left[\frac{(j-1)! + 1}{j} - \left[\frac{(j-1)!}{j} \right] \right],$$

where $[y]$ stands for the largest integer $\leq y$.

(224) Given a sequence of natural numbers $\mathcal{A}$, let $A(n) = \#\{m \leq n : m \in \mathcal{A}\}$, and let us denote respectively by

$$\underline{\mathbf{d}}\,\mathcal{A} = \liminf_{n \to \infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{\mathbf{d}}\,\mathcal{A} = \limsup_{n \to \infty} \frac{A(n)}{n}$$

the *asymptotic lower density* and *asymptotic upper density* of the sequence $\mathcal{A}$. On the other hand, if both these densities are equal, we say that the sequence $\mathcal{A}$ has density $\mathbf{d}\,\mathcal{A} = \underline{\mathbf{d}}\mathcal{A} = \overline{\mathbf{d}}\mathcal{A}$. Prove that:

(a) the density of the sequence made up of all the multiples of a natural number a is equal to $1/a$;

(b) the density of the sequence made up of all the multiples of a natural number a which are not divisible by the natural number a_0 is equal to $\dfrac{1}{a} - \dfrac{1}{[a, a_0]}$;

(c) the density of the sequence made up of all natural numbers which are not divisible by any of the prime numbers $q_1, q_2, \ldots, q_r$ is equal to $\displaystyle\prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right)$.

(225) Let $\mathcal{A}$ be the set of natural numbers n such that $2^{2k} \leq n < 2^{2k+1}$ for a certain integer $k \geq 0$, so that

$$\mathcal{A} = \{1, 4, 5, 6, 7, 16, 17, \ldots, 31, 64, 65, \ldots, 127, 256, 257, \ldots\}.$$

Show that

$$\underline{\mathbf{d}}\mathcal{A} \neq \overline{\mathbf{d}}\mathcal{A}.$$

(226) We say that a sequence of natural numbers $\mathcal{A}$ is *primitive* if no element of $\mathcal{A}$ divides another one. Examples of such sequences are: the sequence of prime numbers, the sequence of natural numbers having exactly k prime factors (k fixed), and finally the sequence of integers n belonging to the interval $]k, 2k]$ (k fixed). Show that if $\mathcal{A}$ is a primitive sequence, then $\overline{\mathbf{d}}\mathcal{A} \leq \frac{1}{2}$.

(227) Let $\mathcal{A}$ be a primitive sequence (see Problem 226). Show that

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} < +\infty.$$

(228) Let $E = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

(a) Show that the sum and the product of elements of E are in E.

(b) Define the norm of an element $z \in E$ by $\|z\| = \|a+b\sqrt{-5}\| = a^2+5b^2$. We say that an element $p \in E$ is *prime* if it is impossible to write $p = n_1 n_2$, with $n_1, n_2 \in E$, $\|n_1\| > 1$, $\|n_2\| > 1$; we say that it is *composite* if it is not prime. Show that, in E, 3 is a prime number and 29 is a composite number.

(c) Show that the factorization of 9 in E is not unique.

(229) Let A be a set of natural numbers and let $A(x) = \#\{n \leq x : n \in A\}$. Show that, for all $x \geq 1$,

$$\sum_{\substack{n \leq x \\ n \in A}} \frac{1}{n} = \sum_{n \leq x} \frac{A(n)}{n(n+1)} + \frac{A(x)}{[x]+1}.$$