The upshot is that, as one way of finding the solution of a linear-equation system \( Ax = d \), where the coefficient matrix \( A \) is nonsingular, is to first find the inverse \( A^{-1} \), and then postmultiply \( A^{-1} \) by the constant vector \( d \). The product \( A^{-1}d \) will then give the solution values of the variables.

**Example 6**

As shown in Example 11 of Sec. 4.2, the simple national-income model

\[
Y = C + Iq + G_0
\]

\[
C = a + bY
\]

can be written in matrix notation as \( Ax = d \), where

\[
A = \begin{bmatrix} 1 & -b \\ -b & 1 \end{bmatrix}, \quad x = \begin{bmatrix} Y \\ C \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} Iq + G_0 \\ a \end{bmatrix}
\]

The inverse of matrix \( A \) is (see explanation in Sec. 5.6)

\[
A^{-1} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}
\]

Thus the solution of the model is \( x^* = A^{-1}d \), or

\[
\begin{bmatrix} Y^* \\ C^* \end{bmatrix} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} Iq + G_0 \\ a \end{bmatrix} = \frac{1}{1-b} \begin{bmatrix} Iq + G_0 + a \\ b(Iq + G_0 + a) \end{bmatrix}
\]

**EXERCISE 4.6**

1. Given \( A = \begin{bmatrix} 0 & 4 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -8 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \), find \( A', B', \) and \( C' \).

2. Use the matrices given in Prob. 1 to verify that
   (a) \( (A + B)' = A' + B' \)
   (b) \( (AC)' = C'A' \)

3. Generalize the result (4.11) to the case of a product of three matrices by proving that, for any conformable matrices \( A, B, \) and \( C \), the equation \( (ABC)' = C'B'A' \) holds.

4. Given the following four matrices, test whether any one of them is the inverse of another:

\[
D = \begin{bmatrix} 1 & 12 \\ 0 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 6 & 8 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad G = \begin{bmatrix} 4 & -\frac{1}{2} \\ -3 & \frac{1}{2} \end{bmatrix}
\]

5. Generalize the result (4.14) by proving that, for any conformable nonsingular matrices \( A, B, \) and \( C \), the equation \( (ABC)^{-1} = C^{-1}B^{-1}A^{-1} \) holds.

6. Let \( A = I - X(X'X)^{-1}X' \).
   (a) Must \( A \) be square? Must \( (X'X) \) be square? Must \( X \) be square?
   (b) Show that matrix \( A \) is idempotent. (Note: If \( X' \) and \( X \) are not square, it is inappropriate to apply (4.14).)

### 4.7 Finite Markov Chains

A common application of matrix algebra is found in what is known as Markov processes or Markov chains. Markov processes are used to measure or estimate movements over time. This involves the use of a Markov transition matrix, where each value in the transition
matrix is a probability of moving from one state (location, job, etc.) to another state. There is also a vector containing the initial distribution across the various states. By repeatedly multiplying such a vector by the transition matrix, one can estimate changes across states over time.

Consider the problem of internal employee movement within a company that has many different branches, or outlets.¹ A simple illustration using two branches, such as Abbotsford and Burnaby, will help to demonstrate the basics of a Markov process. To determine the number of employees in Abbotsford tomorrow, we take the probability that the employees will stay in the Abbotsford branch multiplied by the total number of employees currently in Abbotsford, which gives the total number of current Abbotsford employees who will remain tomorrow. Added to this number is the number of Burnaby employees transferring to Abbotsford. This number is found by multiplying the total number of current Burnaby employees by the probability of a Burnaby employee transferring to Abbotsford. Similarly the process would be the same for determining the number of employees in the Burnaby region tomorrow, made up of those Burnaby employees who chose to remain and the Abbotsford employees who transfer into the Burnaby region today. The process described involves four probabilities. These four probabilities together can be arranged in a matrix. This is known as a Markov transition matrix (or simply, a “Markov”).

Let $A_t$ and $B_t$ denote the populations of Abbotsford and Burnaby, respectively, at some time, $t$. Further, define the transitional probabilities as follows

\[ P_{AA} = \text{probability that a current } A \text{ remains an } A \]
\[ P_{AB} = \text{probability that a current } A \text{ moves to } B \]
\[ P_{BB} = \text{probability that a current } B \text{ remains a } B \]
\[ P_{BA} = \text{probability that a current } B \text{ moves to } A. \]

If we denote the distribution of employees across locations at time $t$ as a vector

\[ x_t = [A_t, B_t] \]

and the transitional probabilities in matrix form

\[ M = \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} \]

then the distribution of employees across locations in the next period ($t+1$) is

\[ [A_t, B_t] \begin{bmatrix} P_{AA} & P_{AB} \\ P_{BA} & P_{BB} \end{bmatrix} = (A_t P_{AA} + B_t P_{BA})(A_t P_{AB} + B_t P_{BB}) \]

\[ = [A_{t+1}, B_{t+1}] \]

¹ We would like to thank Sarah Dunn for this example. This work comes from her final project while a student at the British Columbia Institute of Technology, Burnaby, BC, Canada (June 2003).
To find the distribution of employees after two periods

\[
\begin{bmatrix}
A_{t+1} & B_{t+1}
\end{bmatrix}
\begin{bmatrix}
P_{AA} & P_{AB} \\
P_{BA} & P_{BB}
\end{bmatrix}
= \begin{bmatrix}
A_{t+2} & B_{t+2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_t & B_t
\end{bmatrix}
\begin{bmatrix}
P_{AA} & P_{AB} \\
P_{BA} & P_{BB}
\end{bmatrix}
= \begin{bmatrix}
A_{t+2} & B_{t+2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_t & B_t
\end{bmatrix}
\begin{bmatrix}
P_{AA} & P_{tB} \\
P_{BA} & P_{BB}
\end{bmatrix}^2
= \begin{bmatrix}
A_{t+2} & B_{t+2}
\end{bmatrix}
\]

In general, for \( n \) periods

\[
\begin{bmatrix}
A_t & B_t
\end{bmatrix}
\begin{bmatrix}
P_{AA} & P_{AB} \\
P_{BA} & P_{BB}
\end{bmatrix}^n
= \begin{bmatrix}
A_{t+n} & B_{t+n}
\end{bmatrix}
\]

The 2 \times 2 probability matrix \( M \) is known as the Markov transition matrix. For the case where \( n \) is exogenous, the process is known as a finite Markov chain.

**Example 1**

Suppose the initial distribution of employees across the two locations at time \( t = 0 \) is

\[
\begin{bmatrix}
x_0
\end{bmatrix} = \begin{bmatrix}
A_0 & B_0
\end{bmatrix} = \begin{bmatrix}
100 & 100
\end{bmatrix}
\]

In other words, there are initially equal numbers at each location. Further, let the transitional probabilities in matrix form be as follows:

\[
M = \begin{bmatrix}
P_{AA} & P_{AB} \\
P_{BA} & P_{BB}
\end{bmatrix} = \begin{bmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{bmatrix}
\]

Then the distribution of employees across locations in the next period (\( t = 1 \)) is

\[
\begin{bmatrix}
100 & 100
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{bmatrix} = \begin{bmatrix}
110 & 90
\end{bmatrix} = \begin{bmatrix}
A_1 & B_1
\end{bmatrix}
\]

The distribution after two periods is given by

\[
\begin{bmatrix}
100 & 100
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{bmatrix}^2 = \begin{bmatrix}
100 & 100
\end{bmatrix}
\begin{bmatrix}
0.61 & 0.39 \\
0.52 & 0.48
\end{bmatrix} = \begin{bmatrix}
113 & 87
\end{bmatrix} = \begin{bmatrix}
A_2 & B_2
\end{bmatrix}
\]

The distribution after 10 periods (\( t = 10 \)) is given by

\[
\begin{bmatrix}
100 & 100
\end{bmatrix}
\begin{bmatrix}
0.7 & 0.3 \\
0.4 & 0.6
\end{bmatrix}^{10} = \begin{bmatrix}
100 & 100
\end{bmatrix}
\begin{bmatrix}
0.5174 & 0.4286 \\
0.5174 & 0.4286
\end{bmatrix} = \begin{bmatrix}
114.3 & 85.7
\end{bmatrix} = \begin{bmatrix}
A_{10} & B_{10}
\end{bmatrix}
\]

Notice what happens when the Markov transition matrix is raised to higher and higher powers. The new transition matrix found by raising the original matrix to increasingly higher powers converges to a matrix where the rows are identical. This is referred to as the steady state. What would you expect the eleventh or higher periods of distribution to look like?
Special Case: Absorbing Markov Chains

Now, let us extend the model by adding a third option: Employees can exit the company, with

\[ P_{AE} = \text{probability that a current } A \text{ chooses to exit (E)} \]
\[ P_{BE} = \text{probability that a current } B \text{ chooses to exit (E)} \]

At this point, we will add the following assumptions:

\[ P_{EA} = 0 \quad P_{EB} = 0 \quad P_{EE} = 1 \]

where \( P_{EA}, P_{EB}, \) and \( P_{EE} \) are the probabilities that an employee who is currently an \( E \) will go to \( A, B, \) or \( E, \) respectively. In other words, nobody who leaves the company ever returns. It is also implied by these restrictions that our company never replaces employees that leave (there are no new hires).

Starting at time \( t = 0, \) our Markov chain now becomes

\[
\begin{bmatrix}
A_0 & B_0 & E_0 \\
A_E & B_E & E_E \\
A_E & B_E & E_E
\end{bmatrix}
\begin{bmatrix}
P_{AA} & P_{AB} & P_{AE} \\
P_{BA} & P_{BB} & P_{BE} \\
P_{EA} & P_{EB} & P_{EE}
\end{bmatrix}^n
= \begin{bmatrix}
A_0 \\
B_0 \\
E_0
\end{bmatrix}
\]

(\( A_0, B_0, E_0 \) = 0.)

This type of Markov process is referred to as an absorbing Markov chain. Because of the values of the transition probabilities found in the third row, we see that once an employee becomes an \( E \) in one state (time period) that employee will remain there for all future states (time periods). As \( n \) goes to infinity, \( A_n \) and \( B_n \) will approach zero and \( E_n \) will approach the value of the total number of workers at time zero (i.e., \( A_0 + B_0 + E_0 \)).

EXERCISE 4.7

1. Consider the situation of a mass layoff (i.e., a factory shuts down) where 1,200 people become unemployed and now begin a job search. In this case there are two states: employed (E) and unemployed (U) with an initial vector

\[ X_0 = \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 0 \\ 1200 \end{bmatrix} \]

Suppose that in any given period an unemployed person will find a job with probability .7 and will therefore remain unemployed with a probability of .3. Additionally, persons who find themselves employed in any given period may lose their job with a probability of .1 (and will have a .9 probability of remaining employed).

(a) Set up the Markov transition matrix for this problem.

(b) What will be the number of unemployed people after (i) 2 periods; (ii) 3 periods; (iii) 5 periods; (iv) 10 periods?

(c) What is the steady-state level of unemployment?