Egyptian fractions

Fractions and the notation to represent them were developed in the Middle Kingdom of Egypt, (between 2055 BC and 1650 BC). Five early texts in which fractions appear were: the Egyptian Mathematical Leather Roll, the Moscow Mathematical Papyrus, the Reisner Papyrus, the Kahun Papyrus and the Akhmim Wooden Tablet. A later text, the Rhind Mathematical Papyrus, introduced improved ways of writing Egyptian fractions.

The Rhind papyrus was written by the scribe A’h’mes and dates from the Second Intermediate Period (1650–1550 BC); it includes a table of Egyptian fraction expansions for rational numbers $2/n$, as well as 84 word problems.

Notation

To write the unit fractions used in their Egyptian fraction notation, in hieroglyph script, the Egyptians placed the hieroglyph

(translation: "one among" or possibly "re", mouth) above a number to represent the reciprocal of that number. Similarly in hieratic script they drew a line over the letter representing the number. For example:

$$\begin{align*}
\text{\[image\]} &= \frac{1}{3} & \text{\[image\]} &= \frac{1}{10}
\end{align*}$$

The Egyptians had special symbols for $1/2$, $2/3$, and $\frac{3}{4}$ that were used to reduce fractions larger than $\frac{1}{2}$.

$$\begin{align*}
\text{\[image\]} &= \frac{1}{2} & \text{\[image\]} &= \frac{2}{3} & \text{\[image\]} &= \frac{3}{4}
\end{align*}$$

After subtracting one of these special fractions, the remaining number was written using as a sum of distinct unit fractions.

The reason of this representation is not clear; certainly it's not because it makes the calculation with fractions simpler.

But it is easier to compare two Egyptian fractions than it is to compare two regular fractions because unit fractions compare easily.
Egyptian fraction notation continued to be used in Greek times and into the Middle Ages despite complaints by Ptolemy (AD 90 – AD 168) about the clumsiness of the notation compared to alternatives such as the Babylonian base-60 notation.

**Egyptian fraction**

In number theory, an Egyptian fraction is the sum of distinct unit fractions, such as $\frac{1}{2} + \frac{1}{3} + \frac{1}{16}$. The value of an expression of this type is a positive rational number $a/b$; for instance the Egyptian fraction above sums to $43/48$.

Sums of this type, and similar sums also including $2/3$ and $3/4$ as summands, were used as rational numbers by the ancient Egyptians, and continued to be used by other civilizations into medieval times. In modern mathematical notation, Egyptian fractions have been superseded by regular fractions and decimal notation. However, Egyptian fractions continue to be an object of study in modern number theory and recreational mathematics, as well as in modern historical studies of ancient mathematics.

**Fibonacci’s greedy algorithm for Egyptian fractions**

An important text of medieval mathematics, the Liber Abaci (1202) of Leonardo of Pisa (more commonly known as Fibonacci), provides some insight into the uses of Egyptian fractions in the Middle Ages, and introduces topics that continue to be important in the number theory. He also provides a greedy algorithm to represent every proper reduced fraction (i.e., fractions that are <1 and are such that the denominator and numerator do not have common factors) into an Egyptian fraction. A greedy algorithm is an algorithm that makes a locally optimal choice at each stage of the problem solving process with the hope of finding a global optimum. Fibonacci’s algorithm for transforming rational numbers into Egyptian fractions represents the first published systematic method for constructing such expansions, and it is described in the Liber Abaci (1202).

The general principle behind the algorithm is the following. At each step we fix a “convenient" $d$, and we chose the expansion

$$\frac{x}{y} = \frac{1}{d} + \frac{xd - y}{yd}.$$
d can be any number for which \(xd>y\).

Assuming that \(x<y\) (otherwise the fraction is improper). Let \(q\) be the quotient of the division of \(y\) by \(x\). So, we have \(y=xq+r\), where \(r\) is the remainder of the division.

We chose \(d = q+1\). We write
\[y=xq+r = x(d-1)+r = xd + r-x\]

Thus, \(xd-y = x-r\) and
\[\frac{x}{y} = \frac{1}{d+ \frac{(x-d)/y}{yd}}\]

Note that \(x-r<x\).

We can repeat the procedure with the fraction \((x-r)/yd\) and we continue until all fractions have numerator =1.

This algorithm proves that **Every proper reduced fraction with numerator \(n\) can be written as an Egyptian fraction with \(n\) terms or less.**

**Example.** Write 5/13 as an Egyptian fraction

**Solution.** The quotient of the division of 13 by 5 is \(q=2\), so we chose \(d=3\). Thus.
\[5/13 = 1/3 + (5-3)/(13x3) = 1/3 + 2/39\] To decompose 2/39 in Egyptian fraction we chose \(d=20\).

So, \(2/39 = 1/20 + 1/780\), and
\[5/13 = 1/3 + 1/20 + 1/780\]

This method is effective and easy to apply, but when the numerator is large, it can produce fractions with a very large denominator. Other methods are able to produce Egyptian fractions with smaller denominators, although they may have a large number of terms. In other cases, we get fewer terms and smaller denominators with a “smart” choice of \(d\). For example, we have
Natural questions that arise are: what is the minimum number of (distinct) unit fraction that are necessary to represent a given rational number? And what is the smallest denominator of one such representations?

The Erdős–Straus conjecture

The Erdős–Straus conjecture states that for all integers $n \geq 2$, the rational number $4/n$ can be expressed as the sum of three unit fractions. Paul Erdős and Ernst G. Straus formulated the conjecture in 1948; It is one of many conjectures by P Paul Erdős, a contemporary mathematical genius.

More formally, the conjecture states that, for every integer $n \geq 2$, there exist positive integers $x$, $y$, and $z$ such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

These unit fractions form an Egyptian fraction representation of the number $4/n$.

For instance, for $n = 5$, there are two solutions:

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}.$$

The restriction that $x$, $y$, and $z$ be positive is essential to the difficulty of the problem, for if negative values were allowed the problem could be solved trivially. Also, if $n$ is a composite number, $n = pq$, then an expansion for $4/n$ could be found immediately from an expansion for $4/p$ or $4/q$. Therefore, if a counterexample to the Erdős–Straus conjecture exists, the smallest $n$ forming a counterexample would have to be a prime number.
Computer searches have verified the truth of the conjecture up to $n \leq 10^{14}$, but proving it for all $n$ remains an open problem.

**A short bibliography**

- Tenenbaum, Gérald; Yokota, Hisashi. Length and denominators of Egyptian fractions. (3 papers) (1990)

- M. N. Bleicher, P. Erdős: Denominators of Egyptian fractions, J. Number Theory 8 (1976),