Research Statement by Laura De Carli

For the last ten years I have worked on various problems which are loosely connected with each other. Below I describe some of the most significant and current. The references are to the publications in my CV.

• **Sharp inequalities** An inequality $||Tf|| \leq C||f||$, which is valid for an operator $T$ in a space of functions $f$, is sharp when the constant $C$ cannot be replaced by any smaller constant. Functions $f$ for which $||Tf|| = C||f||$ often solve important problems in Mathematics and Physics. I am especially interested in sharp weighted inequalities for the Fourier transform because of the connections with quantum mechanics and the theory of semigroups. In [14], (probably my best paper), I have proved sharp inequalities for the Hankel transform and stated two conjectures. Two years ago I learned that a group of physicists had independently formulated one of my conjectures and had used the main theorem in [14] to prove uncertainty relation based on Shannon entropies. If my conjecture was proved, they would greatly improve their results. The joint work [24] contains an extension of the main theorem in [14]; in the course of this investigation we have discovered that one of my conjectures is false, but the conjecture that is interesting for the physicists is still valid and challenging.

• **$L^p$ Lebesgue constants** are, roughly speaking, asymptotic estimates of $L^p$ norms of trigonometric polynomials $\sum_{|k| \leq N} e^{ikx}$ in terms of $N$. In dimension $n = 1$ the problem is completely solved and well understood; in dimension $n \geq 2$ there are as many Lebesgue constants as there are generalization of the interval $\{-N, -N + 1, ..., N - 1, N\}$, and their estimates are still not completely understood. The paper [16] (probably my best joint work) deals with Lebesgue constants in several multidimensional cases. Lebesgue constants have interesting connection with Fourier multipliers. An attempt to improve the main theorem in [16] produced results of independent interest. See [26].

• **Bases and frames in Hilbert spaces** play a fundamental role in many applications because they allow to represent functions, often viewed as signals, as a sum of simpler basic frequencies. Frames can be viewed as over-complete bases, and are important in signal processing, control theory, data compression and more. Bases and frames of $L^2(D)$ made of exponential functions are especially relevant for these applications. In [23] we have solved this problem for certain planar domains $D$ and we have proved stability theorems for these bases. Some of the results in [23] are part of A. Kumar’s master project. With my student Z. Hu, I have studied uniqueness properties of tight frames with $n + 1$ elements in $\mathbb{R}^n$. We have recently submitted our joint work [27]

• **Unique continuation for solutions of elliptic PDE** is one of my main research interests since many years ([21], [19], [17], [13], [10], [7], [6], [4],[3]). A typical unique

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1J. Sanchez-Dehesa, (university of Granada ); Lukasz Rudnicki (Polish Academy of Sciences) and others
continuation problem can be roughly described as follows: under which conditions every solution of a partial differential operator which is \( \equiv 0 \) on a certain subset of its domain is \( \equiv 0 \) in the whole domain?

Unique continuation problems have a long history and tradition in Mathematical physics, motivated in part by the study of the spectral properties of the Schrödinger operator \( H = -\Delta + V(x) \) where \( \Delta \) is the Laplacian. It is well known that when \( V \neq 0 \), there may be nontrivial solutions to \( Hu = 0 \) that vanish on the boundary of open sets; however, we have proved in [21] that the support of these solutions cannot be arbitrarily small. This "minimal support property" is enjoyed also by the solutions of other second order elliptic operators. Most of the results in [21] are best possible. Our results generalize the main theorem in [19] (by S. Hudson) and also the well known fact that nontrivial harmonic functions cannot vanish on the boundary of an open and bounded set. Zero sets of harmonic functions have remarkable geometric properties. In [17] we have considered harmonic functions in \( \mathbb{R}^2 \) and extensively investigated the properties of their zeros. Some of our results can be viewed as unique continuation properties for harmonic functions. The conjectures in [17] (by S. Hudson) have attracted considerable interest.\(^3\)

The majority of the unique continuation results in the literature are proved with the aid of Carleman estimates, a tool used also in control theory. My earlier papers are based on Carleman estimates, while in the most recent ones we use new geometric methods. In the recent [25] we prove unique continuation results for the Schrödinger operator \( H = -\Delta + k + V(x) \) where \( k \in \mathbb{R} \), and \( V(x) \in L^\infty \) is supported in a subset of finite measure of the slab \( S = (0,1) \times \mathbb{R}^{n-1} \). We prove that if \( Hu = 0 \) has a nontrivial solution that vanishes on the boundary of \( S \), then the norm of \( V \) cannot be too small. The lower bound for \( ||V||_\infty \) depends on the measure of the support of \( V \) and on the distance of \( k \) and the special set \( \{ m^2 \pi^2 : m \in \mathbb{N} \} \). Various cases are possible.

- **Work in progress and research plans.** The recent conference "Isaac 2013" (Krakov, Aug 5–9) has been a great opportunity to reconnect with colleagues and start new projects: with S. Tikhonov and D. Gorbachov, I have started a new project on Pitt’s inequalities with radial weights. With E. Laeng, I am working on an uncertainty principle for the Hilbert transform.

I am also working on frames and bases on Hilbert spaces: I am planning to prove an extension of a theorems by K. Seip using the ideas in [23]. J. Edward is interested in the application to these results to control theory and hopefully will join me in this project.

I will soon start a new project on finite frame theory with my new student S. Pathak.

\(^3\)see A. Enciso and D. Peralta, Ann. Mat. Pura Appl., DOI 10.1007/s10231-011-0211-4