Solutions to Exam #2

1a) True. This follows from the power law and the differentiation rules \((cf)' = cf'\) and \((f + g)' = f' + g'\).

1b) False. You can’t evaluate the function (i.e. substitute in 2 for \(x\)) before differentiating.

1c) True. We have \(y' = \cos(x)\) and \(y'' = -\sin(x) = -y\).

1d) False. We compute \((\cos(g(x)))'\) by using the chain rule, not the product rule. The correct expression would be \(h'(x) = -\sin(g(x))g'(x)\).

1e) True. Just apply the chain rule.

1f) True. We compute \(y' = 1/x\) so the slope of the tangent line at \((a, \ln(a))\) is \(1/a\) and \(\lim_{a \to 0^+} 1/a = \infty\).

2a) \(\frac{d}{dx} (3x^5 - 2\sqrt{x} + 10^x) = 3(5)x^4 - 2\frac{1}{2\sqrt{x}} + \ln(10)10^x\)

2b) \[\frac{d}{dx} \left( \frac{\arcsin x}{x^2 + 4} \right) = \frac{1}{\sqrt{1-x^2}}(x^2 + 4) - 2x \arcsin(x)}{(x^2 + 4)^2}\]

2c) \[\frac{d}{dx} (e^{\cos x} \tan x) = e^{\cos(x)}(-\sin(x)) \tan(x) + e^{\cos(x)} \sec^2(x)\]

2d) \[\frac{d}{dx} (\ln(\sec(\arctan x))) = \frac{\sec(\arctan(x)) \tan(\arctan(x)) \frac{1}{1+x^2}}{\sec(\arctan(x))} \frac{1}{1+x^2} = \tan(\arctan(x)) \frac{1}{1+x^2} = \frac{x}{1+x^2}\]

2e) Take the logarithm of each side and simplify:

\[\ln(y) = \ln \left( \left( 1 + x^2 \right)^{1/x} \right) = \frac{1}{x} \ln(1 + x^2)\]
We then differentiate both sides:
\[
\frac{dy}{dx} \ln(y) = \frac{d}{dx} \left( \frac{1}{x} \ln(1 + x^2) \right)
\]
\[
y' = \left( -\frac{1}{x^2} \right) \ln(1 + x^2) + \left( \frac{1}{x} \right) \frac{2x}{1 + x^2}
\]
Hence,
\[
y' = y \left( \left( -\frac{1}{x^2} \right) \ln(1 + x^2) + \left( \frac{1}{x} \right) \frac{2x}{1 + x^2} \right)
\]
\[
= (1 + x^2)^{1/x} \left( -\frac{\ln(1 + x^2)}{x^2} + \frac{2}{1 + x^2} \right)
\]

3) Let \( y \) be the altitude of the rocket above the launch pad (in kilometers). Let \( z \) be the distance from the rocket to the radar station. You should make a picture and mark these variables on your picture. Note that both \( y \) and \( z \) vary with time, whereas the horizontal distance between the launch pad and radar station is a constant (30 km). From Pythagorean theorem, \( z^2 = y^2 + (30)^2 \). Differentiate both sides of this equality with respect to \( t \):
\[
2z \frac{dz}{dt} = 2y \frac{dy}{dt}.
\]
When \( z = 50 \), we have \((50)^2 = y^2 + (30)^2\) so \( y = 40 \). Thus, if \( z = 50 \) and \( dz/dt = 60 \), we have
\[
2(50)(60) = 2(40) \frac{dy}{dt},
\]
so \( \frac{dy}{dt} = \frac{2(50)(60)}{2(30)} = \frac{3000}{40} = 75 \) kilometers per minute.

4a) We compute \( f'(x) = \frac{1}{4}x^{-3/4} \). Then \( f(x_0) = f(1) = (1)^{1/4} = 1 \) and \( f'(x_0) = f'(1) = \frac{1}{4}(1)^{-3/4} = \frac{1}{4} \). Hence, the linear approximation is
\[
x^{1/4} \approx 1 + \frac{1}{4}(x - 1).
\]

4b) Using the formula
\[
x^{1/4} \approx 1 + \frac{1}{4}(x - 1),
\]
we see that
\[
(.92)^{1/4} \approx 1 + \frac{1}{4}(0.92 - 1) = 1 + \frac{-0.08}{4} = 0.98.
\]

5) We differentiate implicitly:
\[
4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy').
\]
Now substitute in \( x = 3 \) and \( y = 1 \) into the preceding equation and get:

\[
4(3^2 + 1^2)(2(3) + 2(1)y') = 25(2(3) - 2(1)y')
\]
\[
4(10)(6 + 2y') = 25(6 - 2y')
\]
\[
240 + 80y' = 150 - 50y'
\]
\[
130y' = 150 - 240 = -90
\]
\[
y' = \frac{90}{130} = -\frac{9}{13}
\]

Thus, the slope of the line tangent to the curve at \((3, 1)\) is \(-9/13\) so the equation of the line is:

\[
y - 1 = -\frac{9}{13}(x - 3).
\]

6a) We compute:

\[
\frac{d}{dx} \cos(x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\cos(x) \cos(h) - \cos(x)}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h}
\]
\[
= \cos(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \to 0} \frac{\sin(h)}{h}
\]
\[
= \cos(x)(0) - \sin(x)(1)
\]
\[
= -\sin(x).
\]

6b) We differentiate both sides of the identity \( \cos(\arccos(x)) = x \) and get

\[
\frac{d}{dx} (\cos(\arccos(x))) = \frac{d}{dx} x
\]
\[
- \sin(\arccos(x)) \frac{d}{dx} \arccos(x) = 1
\]

Thus,

\[
\frac{d}{dx} \arccos(x) = -\frac{1}{\sin(\arccos(x))}.
\]

We simplify \( \sin(\arccos(x)) \) by drawing a right triangle containing the angle \( \theta = \arccos(x) \). If the side adjacent to the angle \( \theta \) has length \( x \), then the hypotenuse must have length 1. The side opposite the angle \( \theta \) must then have length \( \sqrt{1 - x^2} \). Hence, \( \sin(\arccos(x)) = \sin(\theta) = \sqrt{1 - x^2}/1 \) and we have

\[
\frac{d}{dx} \arccos(x) = -\frac{1}{\sin(\arccos(x))} = -\frac{1}{\sqrt{1 - x^2}}.
\]
7) The chain rule tells us that

$$h'(x) = f'(g(x))g'(x).$$

We differentiate again and get:

$$h''(x) = \frac{d}{dx} (f'(g(x))g'(x))$$

$$= \left( \frac{d}{dx} (f'(g(x))) \right) g'(x) + f'(g(x)) \frac{d}{dx} g'(x) \quad \text{by the product rule}$$

$$= f''(g(x))g'(x)g'(x) + f'(g(x))g''(x).$$