Mean Value Theorem Worksheet:

1) Consider the function \( f(x) = x(x - 2)^2 \) on the interval \([0, 2]\).

   a) Compute \( f(0) \) and \( f(2) \).

   b) Does Rolle’s theorem apply to \( f(x) \) on the interval \([0, 2]\)?

   c) Find the point(s) \( c \in (0, 2) \) whose existence is guaranteed by Rolle’s theorem.

2) Consider the function \( f(x) = x - \frac{1}{x} \) on the interval \([1, 3]\). Find the point(s) in \((1, 3)\) that are guaranteed to exist by the Mean Value Theorem. Sketch a picture of the graph of \( f(x) \) on \([1, 3]\) including at least one such point and illustrating the Mean Value Theorem.

3) Consider the function \( f(x) = |4x - 8| \) on the interval \([1, 3]\).

   a) Compute \( f(1) \) and \( f(3) \).

   b) Is there \( c \in (1, 3) \) such that \( f'(c) = 0 \)?

   c) Why does Rolle’s Theorem not apply to \( f(x) \) on \([1, 3]\)?

   d) Does the Mean Value theorem apply to \( f(x) \) on the interval \([3, 5]\)? If so find the point(s) \( c \in [3, 5] \) whose existence is guaranteed by the MVT.

4) Suppose that a state police force has deployed an automated radar tracking system on a highway that has a speed limit of 65 mph. A driver passes through one radar detector at 1pm and is traveling 60 mph at that moment. Then, the driver passes through a second radar detector 60 miles away at 1:45pm, again traveling 60 mph at that moment. However, a speeding ticket is being issued for this driver. Argue with Calculus that the speeding ticket is justified.
5) It is intuitively obvious that if $f'(x) > 0$ for all $x \in (a, b)$ then $f(x)$ is increasing. However, the more one thinks about it, this assertion becomes more troubling. The value of $f'(x)$ only controls the slope of the tangent line at the point $x$: why should it say anything about values of $f(x')$ when $x'$ is near $x$? In these problems, we discuss how the Mean Value Theorem gives a rigorous proof of this assertion.

To give such a proof, we need a precise definition of the word *increasing* when it describes a function on an interval. Here is such a definition. We say that a function $f(x)$ is increasing on $(a, b)$ if for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, $f(x_1) < f(x_2)$.

a) If $x_1 < x_2$ what can you say about the sign of $x_2 - x_1$?

b) If $x_1 < x_2$ and
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0,
\]
what can you say about the sign of $f(x_2) - f(x_1)$?

c) In (a) and (b), we have shown that if
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0
\]
then $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Explain how the Mean Value Theorem tells us that if $f'(x) > 0$ for all $x \in (a, b)$ then $f(x)$ is increasing on $(a, b)$. 

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