1. (10 pts) Evaluate the improper integral or show is divergent \( \int_{1}^{+\infty} \frac{1}{1 + x^2} \, dx \) =

\[
= \lim_{b \to +\infty} \int_{1}^{b} \frac{1}{1 + x^2} \, dx
= \lim_{b \to +\infty} \left( \arctan b - \arctan 1 \right)
= \frac{\pi}{2} - \frac{\pi}{4}
= \frac{\pi}{4}
\]

2. (20 pts) Evaluate the integrals (10 pts each):

(a) \( \int x^2 \sin(2x) \, dx \)

\( \begin{align*}
\text{Let } u &= x^2 \sin(2x) \quad \text{Then } du = \sin(2x) \, dx \\
u &= x^2 \sin(2x) \quad \text{and } dv = 2x \, dx \\
u &= x^2 \quad \text{and } v = x^2 \\
\int x^2 \sin(2x) \, dx &= -\frac{1}{2} x^2 \cos(2x) + \int x \cdot 2x \cos(2x) \, dx
\end{align*} \)

(b) \( \int \frac{1}{\sqrt{4 + x^2}} \, dx \)

\( \begin{align*}
\text{Let } u &= x^2 + 4 \quad \text{Then } du = 2x \, dx \\
u &= x^2 + 4 \quad \text{and } dv = 2x \, dx \\
u &= x^2 + 4 \quad \text{and } v = 2x \ln |x + \sqrt{x^2 + 4}| + C
\end{align*} \)

= \ln \left( \frac{1 + x^2}{2} + \frac{x}{2} \right) + C
3. (5 pts) Write the partial fraction decomposition. It is NOT required to determine the constants.
\[
\frac{1}{(x-2)^3(x+2)(x^2+4)^2} = \frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)} + \frac{D}{(x+2)} + \frac{Ex+F}{(x^2+4)} + \frac{Gx+H}{(x^2+4)^2}
\]

4. (15 pts) Circle the correct answer. No justification is necessary for this problem.
(a) Let \( s(t) \) be the position of a particle in rectilinear motion during the time interval \( a \leq t \leq b \). The total distance traveled by the particle is given by

(i) \( \frac{s(b) - s(a)}{b-a} \)  
(ii) \( s(b) \)  
(iii) \( \int_a^b |s'(t)| \, dt \)  
(iv) \( \int_a^b s(t) \, dt \)  
(v) \( s(b) - s(a) \)

(b) The expression \( \frac{d}{dx} \left( \int_0^x \cos(t^2) \, dt \right) \) is equivalent to

(i) \( \sin(x^6) \)  
(ii) \( \cos(x^6) \)  
(iii) \( 6x^5 \cos(x^6) \)  
(iv) \( 12x^2 \cos(x^6) \)  
(v) \( 3x^2 \sin(x^6) \)

(c) Let \( f(x) \) be a continuous function, positive and concave up on the interval \( [a, b] \). Let \( T_6 \) be the trapezoidal approximation with 6 subdivisions of the integral \( \int_a^b f(x) \, dx \). Then compared with the integral, \( T_6 \) is an

(i) overestimate  
(ii) underestimate  
(iii) exact estimate  
(iv) cannot tell (more should be known about \( f \))

(d) The sequence \( a_n = 2 + \frac{(-1)^n}{n} \), \( n \geq 1 \) is

(i) convergent but not monotone  
(ii) monotone but divergent  
(iii) bounded but divergent  
(iv) eventually decreasing but unbounded  
(v) none of the above

(e) The average value of the function \( f(x) \) over the interval \( [a, b] \) is

(i) \( f \left( \frac{a+b}{2} \right) \)  
(ii) \( \frac{f(a) + f(b)}{2} \)  
(iii) \( \frac{a+b}{2} \)  
(iv) \( \frac{f(b) - f(a)}{b-a} \)  
(v) \( \frac{1}{b-a} \int_a^b f(x) \, dx \)

5. (14 pts) Sketch the rose \( r = \sin(3\theta) \) and compute the area of one petal (picture 4pts, computation 10pts).

\[ r = 0 \text{ when } 3\theta = 0, \pi, 2\pi, 3\pi, \ldots, \text{ so we let } \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \ldots \]

The area of one petal is

\[ A_{\text{petal}} = \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/3} \frac{1 - \cos(3\theta)}{2} \, d\theta \]

\[ = \frac{1}{4} \left[ \frac{3}{2} \sin(3\theta) \right]_0^{\pi/3} = \frac{\sqrt{3}}{12} \]

\[ = \frac{1}{4} \left[ \frac{3}{2} \sin(3\theta) \right]_0^{\pi/3} = \frac{\sqrt{3}}{12} \]
6. (14 pts) Find the volume of the solid that results when the region enclosed by \( y = e^{-2x} \), \( y = 0 \), \( x = 0 \) and \( x = 1 \) is revolved about the \( x \)-axis. (Computation is required. Sketch of solid is also required.)

\[ V = \int_0^1 A_{\text{slice}} \cdot \text{Thick} \cdot \text{dx} \]

\[ A_{\text{slice}} = \pi \cdot R^2 \quad \text{where} \quad R = y = e^{-2x} \]

\[ V = \int_0^1 \pi (e^{-2x})^2 \, dx = \frac{\pi}{4} \left[ e^{-4x} \right]_1^0 = \frac{\pi}{4} \left( 1 - \frac{1}{e^4} \right) \]

7. (10 pts) Choose ONE and clearly indicate your choice. Set up the integral only.

(a) A weight of 100 lbs is hanging in a pit 60 feet below ground, suspended (at ground level) by a chain that weighs 0.5 lbs/foot. Set up but do not evaluate an integral that gives the total work to pull the chain and the weight at ground level.

\[ W = \int_0^{60} F(x) \, dx = \int_0^{60} (0.5x + 100) \, dx \]

(b) Set up but do not evaluate an integral that gives the surface area of the surface generated by the curve \( y = \sqrt{x} \), \( 0 \leq x \leq 4 \) when rotated around the line \( x = 4 \).

\[ S = \int_0^4 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

\[ ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \sqrt{1 + \left( \frac{1}{2\sqrt{x}} \right)^2} \, dx \]

\[ y = \sqrt{x} \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \]

\[ R = 4 - x \]

Thus

\[ S = \int_0^4 2\pi (4-x) \sqrt{1 + \frac{1}{4x}} \, dx \]
8. (14 pts) Determine if the series \[ \sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} \] converges. If so, find the sum of the series.

Realize that the series is telescopic

\[ \frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1} \]

(by guess & check or by partial fractions)

So,

\[ S_n = \sum_{k=1}^{n} \left( \frac{1}{2k-1} - \frac{1}{2k+1} \right) = \left( \frac{1}{1} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) \]

Thus,

\[ \sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 1 - \frac{1}{2n+1} \right) = 1 \]

So, the series converges to 1.

9. (20 pts) Is the series absolutely convergent, conditionally convergent, or divergent? Justify in each case.

(a) \[ \sum_{k=2}^{\infty} (-1)^k \sqrt{2/k} \]

Observe that

\[ \lim_{k \to \infty} \sqrt{2/k} = \lim_{k \to \infty} \frac{\sqrt{2}}{\sqrt{k}} = 0 \]

Thus, by the \( k \)-th term test, the given series is divergent.

(b) \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)\sqrt{k}} \]

Test absolute convergence

\[ \sum_{k=2}^{\infty} \left| \frac{(-1)^k}{(k+1)\sqrt{k}} \right| = \sum_{k=2}^{\infty} \frac{1}{(k+1)^{3/2}} \]

But \[ \frac{1}{(k+1)^{3/2}} \leq \frac{1}{k^{3/2}} = \frac{1}{k^{3/2}} \]

and \[ \sum_{k=2}^{\infty} \frac{1}{k^{3/2}} \] is convergent (p-series, with \( p = \frac{3}{2} > 1 \))

Thus \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)\sqrt{k}} \] is convergent, by the simple comparison test.

Thus \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{(k+1)\sqrt{k}} \] is absolutely convergent.
10. (14 pts) Find the radius and the interval of convergence for \[ \sum_{k=1}^{\infty} \frac{(-1)^k(x-1)^k}{3^k \sqrt[k]{k}} \]

**Absolute Ratio Test**

\[
P = \lim_{k \to \infty} \frac{|x-1| \sqrt[k]{k+1}}{3 \sqrt[k]{k+1}} = \lim_{k \to \infty} \left( \frac{|x-1| \sqrt[k]{k+1}}{3 \sqrt[k]{k+1}} \right) = \frac{|x-1|}{3}
\]

\[P < 1 \Rightarrow \frac{|x-1|}{3} < 1 \Rightarrow -3 < x-1 < 3 \Rightarrow \frac{1}{3} < x < 4\]

Thus, the radius of convergence is \( R = 3 \).

Test the end points:

\[x = -2 \quad \sum_{k=1}^{\infty} \frac{(-1)^k(-3)^k}{3^k \sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{1}{2^k} \quad \text{divergent} \quad \rho \text{-series with } \rho = \frac{1}{3} < 1\]

\[x = 4 \quad \sum_{k=1}^{\infty} \frac{(-1)^k4^k}{3^k \sqrt[k]{k}} = \sum_{k=1}^{\infty} \frac{1}{2^k} \quad \text{convergent by A.S.T} \quad \frac{1}{2} = \frac{1}{\sqrt[3]{2}} \xrightarrow{\text{diverges}} 0 \quad \text{so A.S.T applies}
\]

Thus, \( I = (-2, 4] \)

11. (14 pts) (a) (6 pts) Use the Maclaurin series for \( \cos x \) to find a numerical series whose sum is \( \cos 9^\circ \).

\[ \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{with } x \text{ in radians} \]

\[ \cos(9^\circ) = \cos \left( \frac{9^\circ}{180} \pi \right) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{9}{180} \pi)^{2k}}{(2k)!} \]

(b) (8 pts) What is the smallest \( n \) so that the partial sum \( S_n \) of the series in part (a) approximates \( \cos 9^\circ \) with an error less than \( 10^{-4} \)? Be sure to carefully justify your answer. You may use that \( \frac{\pi}{20} < \frac{9}{20} = 1/5 \).

\[ |R_n(x)| \leq \frac{M \cdot |x-x_0|^{n+1}}{(n+1)!} \]

Since the derivatives of \( \cos x \) are \( \pm \sin x \) or \( \pm \cos x \), an upper bound for \( |f^{(n)}(x)| \) is \( M = 1 \).

We want
\[ \frac{\left| \frac{9}{20} \right| - 0}{(n+1)!} < 10^{-4} \]

Try \( n = 3 \).
\[ \left( \frac{\frac{9}{20}}{4!} \right)^4 < \left( \frac{1}{5} \right)^4 = \frac{1}{5^4 \cdot 1234} < \frac{1}{5^4 \cdot 256} < 10^{-4} \]

Thus, \( n = 3 \) works, so

\[ \cos(9^\circ) = \cos(9^\circ) \approx 1 - \frac{1}{100} \]

is the desired.
12. (14 pts) Choose ONE:
   (a) State and prove the geometric series theorem.
   (b) State and prove the the integration formula for area in polar coordinates. A picture, a sum and a limit should appear in your work.

See notes or textbook.