1. Use the definition, to find the Taylor series at \( x_0 = 1 \) for the function \( f(x) = \ln x \). Find the interval of convergence of the series that you obtained. It can be shown that the series converges to \( \ln x \) for all the values of \( x \) in the interval of convergence. Accepting this, what do you obtain when \( x = 2 \)?

Taylor series for \( f(x) \) at \( x_0 = 1 \) is:

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k
\]

where for us \( f(x) = \ln x \) and \( x_0 = 1 \)

\[
f(x) = \ln x \\
f(1) = \ln 1 = 0
\]

\[
f'(x) = \frac{1}{x} \\
f'(1) = 1
\]

\[
f''(x) = -\frac{1}{x^2} \\
f''(1) = -1
\]

\[
f'''(x) = \frac{2}{x^3} \\
f'''(1) = 2
\]

\[
f''''(x) = -\frac{6}{x^4} \\
f''''(1) = -6
\]

\[
f_k(x) = (-1)^{k-1} (k-1)! x^{-k} \\
\Rightarrow f_k(1) = (-1)^{k-1} (k-1)!
\]

So the Taylor series for \( \ln x \) at \( x_0 = 1 \) is:

\[
\ln(1) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (k-1)!}{k!} (x-1)^k
\]

For interval of convergence use absolute ratio test.

\[
P = \lim_{k \to \infty} \left| \frac{f_k(x)}{f_{k+1}(x)} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k-1} (k-1)!}{k!} \cdot \frac{k!}{(-1)^k (k+1)!} \right| = \lim_{k \to \infty} \frac{1}{k+1} = 0
\]

If \( P = |x-1| < 1 \) series absolutely convergent, so this happens when \( 0 < x < 2 \).

Endpoints check: For \( x = 0 \) series becomes:

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1} (k-1)!}{k!} (x-1)^k
\]

For \( x = 2 \), \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \) which converges by A.S.T.

Thus \( I = (0,2) \) and radius of convergence is \( R = 1 \).

Accepting that \( \ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (k-1)!}{k!} (x-1)^k \) for all \( x \in (0,2) \), plugging \( x = 1 \),

in both sides, we get \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \) converges to \( \ln 2 \).