MAC 2311: Worksheet Feb. 16, 2016 (Rules of Differentiation II and Derivatives of Trigonometric Functions)

LECTURE INTRO: Derive the product law.

0) Compute the derivative of
   a) \( \sqrt{x} \sin(x) \):
      \[
      \frac{d}{dx} \left( \sqrt{x} \sin(x) \right) = \frac{1}{2\sqrt{x}} \sin(x) + \sqrt{x} \cos(x)
      \]  
      (Product Rule)
   b) \((x + 2)^2\):
      \[
      \frac{d}{dx} (x^2 + 2x + 4) = 2x + 2
      \]
   c) \(\sqrt[3]{x} \cos(x)\):
      \[
      \frac{d}{dx} \left( x^{\frac{1}{3}} \cos(x) \right) = \frac{1}{3} x^{-\frac{2}{3}} \sin(x) + x^{\frac{1}{3}} (-\sin(x))
      \]

1) For the differentiable function \( f(x) \),
   a) Use the limit definition of derivative to compute the derivative of \( \frac{1}{f(x)} \). This formula is called the Reciprocal Rule.

   \[
   \left( \frac{1}{f(x)} \right)' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \frac{f(x)}{f(x) \cdot f(x) - f(x)}
   \]

   \[
   = - \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \frac{1}{f(x)}
   \]

   \[
   = - f'(x) \cdot \frac{1}{f(x)^2}
   \]

b) Are there any situations where this formula makes no sense?

   1) If \( f \) not differentiable
   2) If \( f(x) = 0 \).
2) Using Product Rule and Reciprocal Rule, we now prove a formula called "Quotient Rule" for the function \( f(x) = \frac{g(x)}{h(x)} \). (Hint: as a first step, observe that \( f(x) = f(x) \cdot \frac{1}{g(x)} \), so we can use Product rule on the right hand. Then use the Reciprocal Rule. Show that the final answer can be written \( (f/g)' = (f'g - gf')/g^2 \).

\[
\left( \frac{f}{g} \right)' = \left( f \cdot \frac{1}{g} \right)' = f' \cdot \frac{1}{g} + f \cdot \left( \frac{1}{g} \right)' = \frac{f'}{g} + f \left( -\frac{g'}{g^2} \right) = \frac{f'g - gf'}{g^2}
\]

3) Compute the following derivatives:

a) \( \frac{d}{dx} \left( \frac{x}{x^2 + 1} \right) = \frac{1 \cdot (x^2 + 1) - x \cdot (2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - x^2}{(x^2 + 1)^2} = \frac{1}{x^2 + 1} \)

b) \( \frac{d}{dx} \left( \frac{\sin(x)}{\sqrt{x}} \right) = \frac{\cos(x) \cdot \sqrt{x} - \frac{1}{2\sqrt{x}} \cdot \sin(x)}{(\sqrt{x})^2} = \frac{\cos(x) \cdot \sqrt{x} - \frac{1}{2} \cdot \sin(x)}{x} \)

c) \( \frac{d}{dt} \tan(t) = \frac{d}{dt} \left( \frac{\sin(t)}{\cos(t)} \right) = \frac{\cos(t) \cdot \cos^2(t) - (\cos^2(t) + \sin^2(t)) \cdot \sin(t)}{\cos^2(t)} = \frac{\cos^2(t) - \sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} = \sec^2(t) \).