(15) 1. Let L be the language accepted by the NFA shown on the right. Find NFAs which accept
   (a) $L^5$
   (b) $(L^5)^R$.

(15) 2 (a) Find an NFA which is equivalent to the RLG given below.
   $G$: $-E$, $E-01E$, $E-1A$, $A-1B$, $A-\lambda$, $B-01$, $B-1C$, $C-E$, $C-0D$, $D-10D$, $D-01$, $D-\lambda$.
   (b) Suppose F is finite and L-F is a regular language. Is it always true that L must also be a regular language? (Justify your answer.)

(15) 3 (a) Write down what the Pumping Lemma says.
   (b) Find a regular expression for the language accepted by the NFA shown on the right.

(22) 4 (a) Define what it means for f to be obtained from g and h by primitive recursion. Show that $f(x,y) = 3x+y+2$ is a primitive recursive function by finding primitive recursive functions g and h such that $f = \text{prim.rec.}(g,h)$.
   (b) Define what it means for f to be obtained from g by minimization. Show that $f(x) = \lceil \sqrt{(2x+1)} \rceil$ is a recursive function by finding a primitive recursive function g such that $f = \mu[g,0]$.
   [You may use the fact that MONUS, ADD, & MULT are prim. rec. if needed in 4(b), but you are not allowed to do so in 4(a).]

(15) 5 (a) Define what is a Turing-semi-decidable relation on N.
   (b) Show what happens at each step if 0100 is the input for the TM, M shown on the right.
   (c) Find the language accepted by M.

(18) 6. Determine which of the following languages are regular and which are not. (a) $L_1 = \{a^k b^n : k \text{mod}\ 3 < n^3 \text{mod}\ 3\}$  (b) $L_2 = \{c^k d^n : k=n^2\}$.
   [If you say that it is regular, you must find a regular expression; if you say it is non-regular, you must give a complete proof.]
2(a) YES. First observe that since \( F \) is finite, \( F \cap L \) will also be finite. Since finite languages are regular, it follows that \( F \cap L \) will be regular. Now \( L = (L - F) \cup (F \cap L) \) and since \( L - F \) is given as regular, it follows by the closure theorem that \( (L - F) \cup (F \cap L) = L \) will be regular.

3(a) Pumping Lemma: Suppose \( L \) is an infinite regular language. Then we can find an \( m \in \mathbb{N} \) such that any string \( w \in L \) with \( |w| \geq m \) can be decomposed as \( w = xyz \) such that \( |xy| \leq m \), \( |y| \geq 1 \), and \( xy^i z \in L \) for each \( i \in \mathbb{N} \).
4(a) \( f \) is obtained from \( g \) \& \( h \) by primitive recursion if
\[
\begin{align*}
f(x,0) &= g(x) \quad \text{and} \quad f(x, y+1) = h(x, y, f(x, y)).
\end{align*}
\]
Here \( x = \langle x_1, \ldots, x_n \rangle \), \( g: \mathbb{N}^n \to \mathbb{N} \), \( h: \mathbb{N}^{n+2} \to \mathbb{N} \) \& \( f: \mathbb{N}^n \to \mathbb{N} \).

\[
f(x, y) = 3x + y + 2. \quad \text{So} \quad f(x, 0) = 3x + 2 \Rightarrow g(x) = 3x + 2.
\]

Also \( f(x, y+1) = 3(x + (y+1)) + 2 = (3x + y + 2) + 1 = f(x, y) + 1 \)
\[
\Rightarrow h(x, y, f(x, y)) = f(x, y) + 1.
\]
\[
h = \text{so } I_3^{(a)}.
\]

Now \( g(0) = 2 = \text{so } so_0 \) \& \( g(y+1) = 3(y+2) + 2 = 3y + 3 = g(y) + 3 \)
\[
\Rightarrow g = \text{prim. rec. } (\text{so } so_0, \text{so } so_0 \text{ } I_2^{(a)}).
\]

So
\[
f = \text{prim. rec. } (g, h) = \text{prim. rec. } (\text{prim. rec. } (\text{so } so_0, \text{so } so_0 \text{ } I_2^{(a)}), \text{so } I_3^{(a)}),
\]
and hence \( f \) is a primitive recursive function.

(b) \( f \) is obtained from \( g \) by minimization if \( f: \mathbb{N}^n \to \mathbb{N} \)
\[
\& \quad f(x) = \text{smallest } y \text{ such that } g(x, y) = 0
\]
\[
\text{undefined if } g(x, y) \neq 0 \text{ for every } y.
\]
Here \( x = \langle x_1, \ldots, x_n \rangle \) \& \( g: \mathbb{N}^{n+1} \to \mathbb{N} \) is a total function.
4(b) Let \( g(x,y) = 2x+1 = y^2 \). Then \( (\mu y)[g(x,y) = 0] \)

\[
= (\mu y)[2x + 1 = 0] = \sqrt{2x + 1} = f(x).
\]

\( : f = \mu [g, \mu] = \mu [\text{MONUS} \circ \text{ADD} \circ [I_1^{(b)}, I_1^{(b)}], \text{MULT} \circ [I_2^{(b)}, I_2^{(b)}], 0] \)

and so \( f \) is a recursive function.

5(a) The relation \( R \) is Turing semi-decidable if we can find a TM \( M \) which halts in an accepting state whenever \( \langle m, n \rangle \in R \); and which fails to halt or halts in a non-acc. state when \( \langle m, n \rangle \notin R \).

(b) \( \langle C, 0100 \rangle \rightarrow \langle D, 0100 \rangle \rightarrow \langle E, 0000 \rangle \rightarrow \langle D, 0010 \rangle \rightarrow \langle G, 0010 \rangle \rightarrow \langle F, 0010 \rangle \).

(c) \( L(M) = 1.0^* + 0(0.0)^*0.0^* + 0.1(0.1)^* \).

6(a) \( n \equiv 0 \pmod{3} \Rightarrow k \pmod{3} = 0 \) (not poss.), \( n \equiv 1 \pmod{3} \Rightarrow k \pmod{3} = 1^2 \)

\( n \equiv 2 \pmod{3} \Rightarrow k \pmod{3} = 2^2 \pmod{3} \Rightarrow k \equiv 0 \pmod{3} \)

\( : L_1 = (aaa)^*(bbb)^*b + (aaa)^*(bbb)^*bb \) is a regular lang.

(b) \( L_2 = \{c^k d^n : k < n^2 \} \). Supp. \( L_2 \) was regular. Then we can find an NFA \( M \) such that \( L(M) = L_2 \). Let \( N \) = the number of states in \( M \). Then \( c^{N^2-1} d^N \in L_2 \) because if we put \( k = N^2 - 1 \) \& \( n = N \) we will see that \( k < n^2 \). So \( M \)

will accept \( c^{N^2-1} d^N \). Since it takes \( NH \) states to process the \( d^N \), the acceptance track of \( c^{N^2-1} d^N \) must have a cycle as shown below, with \( j > 1 \).

\[
\begin{array}{c}
\rightarrow \ 0_0 \\
\rightarrow \ 0_{N^2-1} \\
\end{array}
\]

Now if we skip the cycle, we will see that \( M \) accepts the string \( c^{N^2-1} d^e d^{N-1-j} = c^{N^2-1} d^{N-j} \). But \( c^{N^2-1} d^{N-j} \)

is not in \( L_2 \) because \( N^2 - 1 < (N-j)^2 \). So this contradicts the fact that \( L(M) = L_2 \). Hence \( L_2 \) is non-regular.