Answer all 6 questions. A neat and clear presentation is essential for full credit. Show all working and provide all reasoning. An unjustified answer will receive little or no credit.

(16) 1. Find the solution of the equation $a_{n+2} + 6a_{n+1} + 9a_n = 0$ with the initial conditions $a_0 = 1$, $a_1 = 3$.

(24) 2. Find the general solution of the following difference equations
   
   (a) $a_{n+2} - 2a_{n+1} + 5a_n = 8$
   
   (b) $a_{n+2} + 2a_{n+1} - 3a_n = 12$.

(20) 3. (a) Let $D_n$ be the no. of derangements of $\{1,2,3,\ldots,n\}$ and let $E_n$ be the no. of derangements of $\{1,2,3,\ldots,n\}$ in which the first element is 2. Prove that $E_n = D_{n-1} + D_{n-2}$.
   
   (b) Hence prove that $D_n = (n-1)(D_{n-1} + D_{n-2})$.

(20) 4. Use the method of generating functions to find the solution of the difference equation $a_n - a_{n-1} - 2a_{n-2} = 0$ with the initial conditions $a_0 = 1$ and $a_1 = 8$.

(20) 5. (a) Starting with $(1-x)^{-1} = 1 + x + x^2 + \cdots + x^n + \cdots$, find the generating function for $<3n/2^n>_{n\geq 0}$.

   (b) Let $S = [\omega, a, \omega, b, \omega, c]$ and $h_n$ be the no. of $n$-combinations of $S$ in which the no. of $a$'s is odd, the no. of $b$'s is a multiple of 3, and the no. of $c$'s is even and $\geq 4$. Find the generating function of $<h_n>_{n\geq 0}$.

   (Give your answers as functions, not as infinite series.)

(20) 6. (a) Define what are the standard generating function and the exponential generating function of a sequence $<h_n>_{n\geq 0}$.

   (b) If $n \geq 2$, prove that in any group of $n$ people we can always find 2 people who have the same number of friends in the group.
1. 

\[ a_{n+2} + 6a_{n+1} + 9a_n = 0 \]
\[ (E^2 + 6E + 9) a_n = 0 \]

Aux. eq. is  
\[ E^2 + 6E + 9 = 0 \]
\[ (E+3)^2 = 0 \]

\[ \therefore E = -3 \text{ (twice)} \]

So  
\[ a_n = (A + nB) \cdot (-3)^n \]

Since  \[ a_0 = 1 \]  &  \[ a_1 = 3 \]  we have  

\[ a_0 = 1 = (A + 0B) \cdot (-3)^0 \]
\[ a_1 = 3 = (A + 1B) \cdot (-3)^1 \]

\[ \therefore 1 = A \]
\[ 3 = (A+B) \cdot (-3) \Rightarrow A+B = -1 \]

\[ \therefore A = 1 \]  &  \[ B = -1-1 = -2 \]

So  
\[ a_n = (1-2n) \cdot (-3)^n . \]

2. (a) 

\[ a_{n+2} - 2a_{n+1} + 5a_n = 0 \]
\[ (E^2 - 2E + 5) a_n = 0 \]

\[ E^2 - 2E + 5 = 0 \]

\[ \therefore E = \frac{-( -2) \pm \sqrt{4 - 4.15}}{2} \]

\[ = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i \]

\[ \therefore a_n^c = A \cdot (1+2i)^n + B \cdot (1-2i)^n \]
2(a) (cont.) Try \( a_n^p = b \). Then
\[
\begin{align*}
a_{n+1}^p &= b \\
a_{n+2}^p &= b
\end{align*}
\]

So \( a_{n+2} - 2a_{n+1} + 5a_n = 8 \) becomes
\[
4b = 8 \implies b = 2.
\]

So \( a_n^p = 2 \)

\[
\therefore a_n = a_n^c + a_n^p = A \cdot (i+2i)^n + B \cdot (1-2i)^n + 2
\]
is the general solution.

(b) \( a_{n+2} + 2a_{n+1} - 3a_n = 0 \)

\[
E^2 + 2E - 3 = 0 \quad \text{(aux. eq.)}
\]

\[
(E + 3)(E - 1) = 0 \implies E = -3 \text{ or } 1
\]

\[
\therefore a_n^c = A \cdot (-3)^n + B \cdot (1)^n = A + B \cdot (-3)^n
\]

Since \( 1 \) is root of multiplicity 1 in the aux. eq.
we must try \( a_n^p = b \cdot n = b \cdot n \)

\[
\begin{align*}
a_{n+1}^p &= b \cdot (n+1) \\
a_{n+2}^p &= b \cdot (n+2)
\end{align*}
\]

So \( a_{n+2} + 2a_{n+1} - 3a_n = 12 \) becomes

\[
\begin{align*}
(b \cdot n+2) + 2 \cdot b \cdot (n+1) - 3 \cdot b \cdot n &= 12 \\
(b + 2b - 3b) \cdot n + 2b + 2b &= 12
\end{align*}
\]

\[
\therefore 4b = 12 \implies b = 3
\]

\[
\therefore a_n^p = 3n
\]

So \( a_n = a_n^c + a_n^p = A + B \cdot (-3)^n + 3n \).
3(a) Consider an arbitrary derangement in $E_n$.

Case (i) $\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}$

Case (ii) $\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{pmatrix}$

Either 2 goes to 1 or 2 goes to one of the elements 3, ..., $n$. In the first case 1 goes to 2 & 2 goes to 1 and the elements 3, ..., $n$ must be deranged. Since there are $D_{n-2}$ ways of deranging $n-2$ elements this contributes $D_{n-2}$ derangements to $E_n$. In the second case 2 does not go to 1 — so if we pretend 1 in 2, this would be equivalent to deranging 2, 3, ..., $n$. In other words, the second case will provide $D_{n-1}$ derangements in $E_n$. Hence

$$E_n = D_{n-2} + D_{n-1} = D_{n-1} + D_{n-2}.$$ 

(b) Now in part (a) we considered that 2 went to 1. But we could have had any element 3, ..., $n$ going to 1 (we can't have 1 going to 1 because we want a derangement of 1, ..., $n$). Since there were $n-1$ choices for the first element and each choice [such as the 2 in part (a)] produced $D_{n-1} + D_{n-2}$ derangements, the total number of derangements of 1, 2, ..., $n$ must be $(n-1) \cdot (D_{n-1} + D_{n-2})$. Hence

$$D_n = (n-1) \cdot (D_{n-1} + D_{n-2}).$$
4. Let \( f(x) \) be the generating function of \( \langle a_n \rangle_{n=0}^{\infty} \).

Then
\[
\begin{align*}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \\
-x f(x) &= a_0 x - a_1 x^2 - \cdots - a_{n-1} x^n - \cdots \\
-2x^2 f(x) &= -2a_0 x^2 - \cdots - 2a_{n-2} x^n - \cdots
\end{align*}
\]

Adding these equations we get
\[
(1 - x - 2x^2) f(x) = a_0 + (a_1 - a_0) x + (a_2 - a_1 - 2a_0) x^2 + \cdots + (a_n - a_{n-1} - 2a_{n-2}) x^n + \cdots
\]
\[
= a_0 + (a_1 - a_0) x + 0 x^2 + \cdots + 0 x^n + \cdots
\]
\[
= 1 + (8-1)x = 1 + 7x
\]

\[
\therefore f(x) = \frac{1 + 7x}{1 - x - 2x^2} = \frac{1 + 7x}{(1 + x)(1 - 2x)}
\]

Let \( \frac{1 + 7x}{(1 + x)(1 - 2x)} = \frac{A}{1 + x} + \frac{B}{1 - 2x} \)

Then \( 1 + 7x = A(1 - 2x) + B(1 + x) \)

Putting \( x = -1 \) gives us
\[
1 - 7 = A[1 - (-2)] + B \cdot 0 \quad \Rightarrow -6 = 3A \quad \Rightarrow A = -2
\]

Putting \( x = \frac{1}{2} \) gives us
\[
1 + 7/2 = A \cdot 0 + B \cdot \frac{3}{2} \quad \Rightarrow \frac{9}{2} = \frac{3B}{2} \quad \Rightarrow B = 3
\]

\[
\therefore f(x) = \frac{-2}{1 + x} + \frac{3}{1 - 2x} = (-2) \cdot [1 + (-x) + (-x)^2 + \cdots + (-x)^n + \cdots] + 3 \cdot [1 + 2x + 2^2 x^2 + \cdots + 2^n x^n + \cdots]
\]

\[
\therefore a_n = \text{coefficient of } x^n = -2 \cdot (-1)^n + 3 \cdot 2^n.
\]
5(a) \[ 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x} \]

Differentiating both sides we get
\[ 0 + 1 + 2x + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2} \]

Multiplying both sides by 3x we get
\[ 0 + 3x + 3.2x^2 + \cdots + 3n.x^n + \cdots = \frac{3x}{(1-x)^2} \]

Replacing \( x \) by \( x/2 \) we get
\[ 0 + 3 \cdot \frac{x}{2} + 3.2 \cdot \left(\frac{x}{2}\right)^2 + \cdots + 3n.\left(\frac{x}{2}\right)^n + \cdots = \frac{3x/2}{(1-x/2)^2} \]

\[ \therefore \sum_{n=0}^{\infty} 3n.\frac{1}{2^n} \cdot x^n = \frac{3}{2} \cdot \frac{x}{(1-x/2)^2} \]

So gen. func. of \( \left\langle \frac{3n}{2^n} \right\rangle_{n=0}^{\infty} \) is \( \frac{3}{2} \frac{x}{(1-x/2)^2} \)

(b) \( h_n = \) coefficient of \( x^n \) in the expansion of
\[ (x+x^3+x^5+\cdots)(1+x^3+x^6+\cdots)(x^4+x^6+x^8+\cdots) \]
\[ = x(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots) \cdot x^4(1+x^2+x^4+\cdots) \]
\[ = x^5 \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^2} \]

\[ \therefore \text{ generating function of } \left\langle h_n \right\rangle_{n=0}^{\infty} \text{ is} \]
\[ \frac{x^5}{(1-x^2)^2 \cdot (1-x^3)} \]
6. (a) The standard generating function of \(\{h_n\}_{n=0}^{\infty}\) is the function
\[
    f(x) = \sum_{n=0}^{\infty} h_n \cdot x^n
\]
The exponential generating function of \(\{h_n\}_{n=0}^{\infty}\) is the function
\[
    g(x) = \sum_{n=0}^{\infty} \frac{h_n \cdot x^n}{n!}
\]

(b) The proof splits into two cases.

Case (i): There is a person in the group who has no friends in the group.
In this case the maximum number of friends a person can have is \(n-2\). So the possibilities are \(0, 1, 2, \ldots, n-2\) friends. Since there are \(n\) people \& \(n-1\) possibilities, two people must have the same number of friends by the Pigeon Hole Principle.

Case (ii): There is no person in the group who has no friends in the group.
In this case the possible number of friends a person can have are \(1, 2, 3, \ldots, n-1\).
Since there are \(n\) people \& \(n-1\) possibilities we must have two people, again, who have the same number of friends by the Pigeon Hole principle.