Answer all 6 questions. An unjustified answer or failure to follow instructions will result in little credit. So show all working and provide all reasoning. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(16) 1. Find the solution of the equation $a_{n+2} + 6a_{n+1} + 9a_n = 0$ with the initial conditions $a_0=4$, $a_1=3$.

(24) 2. Find the general solution of the following difference equations
   
   (a) $a_{n+2} - 2a_{n+1} - 3a_n = 8$.
   (b) $a_{n+1} - 3a_n = 12.3^n$.

(20) 3. (a) Define the Stirling numbers of the First kind and the Stirling numbers of the Second kind.

   (b) Let $h_n = 6n^2-4n+3$. So $\langle h_n \rangle = \langle 3, 5, 19, 45, 83, 133, \ldots \rangle$. Find the zero diagonal of $\langle h_n \rangle$ and a formula for the sum $h_0+h_1+h_2+\ldots+h_n$. (Simplify your answer)

(20) 4. Use the method of generating functions to find the solution of the difference equation $a_n + 3a_{n-1} + 8 = 0$ with the initial condition $a_0=3$.

(20) 5. (a) Starting with $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^n + \ldots$, find the generating function for $\langle (n+1)/(-2)^n \rangle_{n \geq 0}$.

   (b) Let $M = [2.a, 3.b, 3.c]$. Use the method of exponential generating functions to find the number of 5 -permutations of M. (Express your answers in terms of factorials and simplify as far as possible)

(20) 6. (a) Define what is the zero diagonal (=zero column) of a sequence $\langle h_n \rangle_{n \geq 0}$.

   (b) Prove that in any group of 10 people we can always find 3 mutual friends or 4 mutual strangers. (You may use the fact that in any group of 6 people we can always find 3 mutual friends or 3 mutual strangers, if needed.)
1. \[ a_{n+2} + 6a_{n+1} + 9a_n = 0 \]
\[ (E^2 + 6E + 9)A_n = 0 \]
Aux. Eq. \[ E^2 + 6E + 9 = 0 \]
\[ (E+3)(E+3) = 0 \]
\[ \therefore E = -3 \text{ (twice)} \]
\[ \therefore a_n = (A + Bn)(-3)^n \]

But \[ a_0 = 4 \text{ and } a_1 = 3. \]
So \[ 4 = (A + 3)(-3)^0 \implies A = 4 \]
\[ 3 = (A + B)(-3)^1 \implies 3 = -3(A + B) \]
\[ \implies 1 = -(4 + B) \implies B = -1 - 4 = -5. \]
\[ \therefore a_n = (4 - 5n)(-3)^n. \]

2(a) \[ a_{n+2} - 2a_{n+1} - 3a_n = 8 \quad (**) \]
Homog. Eq. \[ a_{n+2} - 2a_{n+1} - 3a_n = 0 \quad (*) \]
\[ (E^2 - 2E - 3) = 0 \]
\[ (E+1)(E-3) = 0 \]
\[ \therefore (a_n) = A(1)^n + B(3)^n \]

Try \( (a_n)_p = b. \) Then \( (a_{n+1})_p = b \) \& \( (a_{n+2})_p = b \)
So \( (**) \) becomes \[ b - 2b - 3b = 8 \]
\[ -4b = 8 \quad \text{so} \quad b = -2 \]
\[ \therefore a_n = (a_n)_c + (a_n)_p = A(1)^n + B(3)^n - 2. \]
2 (b) \( a_{n+1} - 3a_n = 12 \cdot (3)^n \) \((**)\)

Homog. Eq.: \( a_{n+1} - 3a_n = 0 \)

Aux. Eq.: \( E - 3 = 0 \Rightarrow E = 3 \)

\( \therefore (a_n)_c = A \cdot (3)^n \).

Since 3 is a root of the aux. eq., try \((a_n)_p = b \cdot n \cdot (3)^n \).

\((a_{n+1})_p = b \cdot (n+1) \cdot 3^{n+1} = (3b + 3bn) \cdot 3^n \)

So \((***)\) becomes \((3bn + 3b) \cdot 3^n - 3 \cdot (bn \cdot 3^n) = 12 \cdot 3^n \)

\( \therefore 3b \cdot 3^n = 12 \cdot 3^n \)

\( \therefore b = 4 \).

So \( a_n = (a_n)_c + (a_n)_p = A \cdot 3^n + 4n \cdot 3^n \)

3. (a) The Stirling numbers of the 1st kind are the unique integers \( \{p\} \) such that

\( [n]_p = \sum_{k=0}^{p} (-1)^{p-k} \cdot \{p\} \cdot n^k \)

The Stirling numbers of the 2nd kind are the unique integers \( \{p^2\} \) such that

\( n^p = \sum_{k=0}^{p} \{p^2\} \cdot [n]_k \)

(b) \( h_n = 6n^2 - 4n + 3 \)

\[ \begin{array}{ccccccc}
\Delta^0 h_n & | & 3 & 5 & 19 & 45 & 83 & 133 \\
\Delta h_n & | & 2 & 14 & 26 & 38 & 50 & \\
\Delta^2 h_n & | & 12 & 12 & 12 & 12 & \\
\Delta^3 h_n & | & 0 & 0 & 0 & & \\
\end{array} \]
3(b) So zero column = \( <3, 2, 12, 0, \ldots> \)

\[
\therefore h_k = 3 \binom{k}{0} + 2 \binom{k}{1} + 12 \binom{k}{2}
\]

So \( \sum_{k=0}^{n} h_k = 3 \binom{n+1}{0} + 2 \binom{n+1}{1} + 12 \binom{n+1}{2} \)

\[
= 3 \frac{(n+1)}{1!} + 2 \frac{(n+1)(n)}{2!} + 12 \frac{(n+1)(n)(n-1)}{3!}
\]

\[
= 3(n+1) + (n+1)n + 2(n+1)(n^2 - n)
\]

\[
= (n+1)(3 + n + 2n^2 - 2n)
\]

\[
= (n+1)(2n^2 - n + 3)
\]

4. Let \( f(x) = \text{generating function of } \{a_n\}_{n=0}^{\infty} \). Then

\[
f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots
\]

\[
3x f(x) = 3a_0 x + 3a_1 x^2 + \ldots + 3a_{n-1} x^n + \ldots
\]

\[
\frac{8}{1-x} = 8 + 8x + 8x^2 + \ldots + 8x^n + \ldots
\]

\[
\therefore (1+3x) f(x) + \frac{8}{1-x} = (a_0 + 8) + (a_1 + 3a_0 + 8)x + (a_2 + 3a_1 + 8)x^2 + \ldots
\]

\[
= (3+8) + 0.x + 0.x^2 + \ldots + 0.x^n + \ldots
\]

\[
\therefore (1+3x) f(x) = 11 - \frac{8}{1-x} = \frac{3 - 11x}{1-x}
\]

\[
\therefore f(x) = \frac{3 - 11x}{(1-x)(1+3x)} = \frac{A}{1-x} + \frac{B}{1+3x}
\]

\[
\therefore 3 - 11x = A(1-x) + B(1+3x)
\]

Putting \( x = -\frac{1}{3} \) gives us \( 3 + \frac{11}{3} = A \cdot \frac{4}{3} \)

\[
\therefore \frac{20}{3} = 4A \Rightarrow A = 5.
\]
4. Putting $x = 1$ gives us $3 - 11 = B \cdot (1 + 3)$
\[
-8 = 4B \quad \Rightarrow \quad B = -2
\]
\[
\therefore \quad f(x) = \frac{5}{1 + 3x} + \frac{-2}{1 - x} = \frac{5}{1 - (-3x)} + \frac{-2}{1 - x}
\]
\[
= 5 \left[ 1 + (3x) + (3x)^2 + \ldots + (3x)^n + \ldots \right] - 2 \left[ 1 + x + x^2 + \ldots + x^n + \ldots \right]
\]
\[
\therefore \quad a_n = \text{coeff. of } x^n \text{ in the expansion of } f(x)
\]
\[
= 5 \cdot (-3)^n - 2.
\]

5(a) \[
1 + x + x^2 + \cdots + x^n + \cdots = (1 - x)^{-1}
\]
Differentiating both sides we get
\[
0 + 1 + 2x + 3x^2 + \cdots + n x^{n-1} + (n+1)x^n + \cdots = (-1)^2 (1 - x)^{-2}
\]
\[
\therefore \quad 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots = (1 - x)^{-2}
\]
Replacing $x$ by $-\frac{x}{2}$ we get
\[
1 + 2 \cdot \left(-\frac{x}{2}\right) + 3 \cdot \left(-\frac{x}{2}\right)^2 + \cdots + \frac{n+1}{(-2)^n} \cdot x^n + \cdots = \left(\frac{1 + x}{2}\right)^{-2}
\]
\[
= \frac{4}{(2 + x)^2}
\]
\[
\therefore \quad \text{gen. function of } \sum_{n=0}^{\infty} \frac{n+1}{(-2)^n} \text{ in } \frac{4}{(2 + x)^2}.
\]

(b) Number of 5-permutations of $[2, a, 3, b, 3, c] = M$ is the coefficient of $x^5$ in the factorial expansion of
\[
\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)
\]
\[
\frac{1}{1!} \cdot \frac{x^2}{2!} \cdot \frac{x^3}{3!} + \frac{1}{1!} \cdot \frac{x^3}{2!} \cdot \frac{x^2}{3!} + \frac{1}{1!} \cdot \frac{x^3}{3!} \cdot \frac{x^2}{2!} + \frac{1}{1!} \cdot \frac{x^3}{2!} \cdot \frac{x^2}{2!} + \frac{1}{1!} \cdot \frac{x^3}{3!} \cdot \frac{x^2}{2!}
\]
\[
\therefore \quad \text{Ans} = 5\left(\frac{4}{2! \cdot 3!} + \frac{2}{3!} + \frac{3}{2! \cdot 2!}\right) \cdot 5!.
6. (a) The zero column of \( \langle h_n \rangle_{n=0}^{\infty} \) is defined to be the sequence \( \langle \Delta^k h_0 \rangle_{k=0}^{\infty} = \langle \Delta h_0, \Delta^2 h_0, \Delta^3 h_0, \ldots, \Delta^k h_0, \ldots \rangle \)

(b) Choose one of the 10 people & call her \( p_i \). Let \( F = \) friends of \( p_i \) & \( S = \) strangers to \( p_i \). Then \( |FU_S| = 9 \) and \( FNS = \emptyset \). So either \( |F| > 4 \) or \( |S| \geq 6 \).

Case(i) \( |F| > 4 \): In this case either there are 2 mutual friends in \( F \) or everyone in \( F \) are mutual strangers.

Case(ii) \( |S| \geq 6 \): In this case we know that there are 3 mutual friends or 3 mutual strangers in \( S \) by a theorem in class. If there are 3 mutual friends then we got our 3 mutual friends. And if there are 3 mutual strangers in \( S \), we can add \( p_i \) to get 4 mutual strangers.

So in both cases we get 3 mutual friends or 4 mutual strangers.

5. (b) Alternative solution.

\[ \text{Ans} = \text{coeff. of } x^5 \text{ in the expansion of} \]

\[ \left( 1 + \frac{x}{11} + \frac{x^2}{21} \right) \left( 1 + \frac{x}{11} + \frac{x^2}{21} + \frac{x^3}{31} \right) \left( 1 + \frac{x}{11} + \frac{x^2}{21} + \frac{x^3}{31} + \frac{x^4}{37} \right) \]

\[ = \left( 1 + \frac{x}{2} \right) \left( 1 + 2x + 2x^2 + \frac{2}{3} x^3 + \left( \frac{1}{4} + \frac{1}{6} \right) x^4 + \frac{1}{12} x^5 + \cdots \right) \]

\[ = 1 + \cdots + \left[ \frac{1}{2} \left( \frac{4}{3} + \frac{1}{12} + \frac{1}{4} \right) + \frac{1}{12} \right] x^5 + \cdots \]

\[ = 1 + \cdots + \left( \frac{17}{12} \cdot 5! \right) \frac{x^5}{51} + \cdots \]

So \( \text{Ans} = \frac{17}{12} \cdot 5! = \frac{17}{12} \cdot 120 = 170 \)