1. Starting with \( \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \), find generating functions for the sequences \((-2)^n\) \(\forall n \geq 0\) and \(\langle a_n \rangle \), \(\forall n \geq 0\).

2. Find the general solution of the following difference equations:
   (a) \(x_{n+2} - 2x_{n+1} + 4x_n = 0\)
   (b) \(4x_{n+2} + 4x_{n+1} + x_n = 3n\)

3. Find the solution of the equation
   \(x_{n+2} + x_{n+1} - 6x_n = 4\)
   with the initial conditions \(x_0 = 3\), \(x_1 = 2\).

4. By using generating functions, find the solution of the equation
   \(a_n - 2a_{n+1} + 1 = 0\)
   with initial condition \(a_0 = 0\).

5. Let \(G\) be a connected planar graph with \(n\) vertices and \(e\) edges which divides the plane into \(f\) regions. If \(G\) has no cycles of length 3, 4 or 5 show that:
   (a) \(3f \leq e\)
   (b) \(2e \leq 3n - 6\)

6. Let \(G\) be a graph with \(n\) vertices. Prove that \(G\) must have two vertices with the same degree.
1. We know that \[ \frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \quad (*) \]

So, \[ \frac{1}{1-(-2x)} = 1 + (-2x) + (-2x)^2 + \ldots + (-2x)^n + \ldots \]

\[ = 1 - 2x + 2^2x^2 + \ldots + (-2)^n x^n + \ldots \]

the generating function of \( \langle (-2)^n \rangle_{n=0}^\infty \) is

\[ \frac{1}{1-(-2x)} = \frac{1}{1+2x} \]

Also by differentiating both sides of (*) we obtain

\[ \frac{-1}{(1-x)^2} \cdot (-1) = 0 + 1 + 2x + 3x^2 + \ldots + nx^n + \ldots \]

\[ \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots + nx^n + \ldots \]

Multiplying both sides by \( 2x \) gives us

\[ \frac{2x}{(1-x)^2} = 0 + 2x + 3x^2 + \ldots + 2nx^n + \ldots \]

the generating function of \( \langle 2n \rangle_{n=0}^\infty \) is \( \frac{2x}{(1-x)^2} \)

2. (a) \[ x_{n+2} - 2x_{n+1} + 4x_n = 0 \]

Aux. eq. \[ E^2 - 2E + 4 = 0 \]

\[ \therefore E = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2} \]

\[ = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}i \]

So the general solution is

\[ x_n = A \cdot (1+i \sqrt{3})^n + B \cdot (1-i \sqrt{3})^n \]
2. (b) \[ 4x_{n+2} + 4x_{n+1} + x_n = 3n \quad \ldots \quad (*) \]

Homog. eq is: \[ 4x_{n+2} + 4x_{n+1} + x_n = 0 \]

Aux. eq is: \[ 4E^2 + 4E + 1 = 0 \]

\[ (2E+1)(2E+1) = 0 \]

So \[ E = -\frac{1}{2} \quad \text{(twice)} \]

The complementary solution is thus \[ x_n^c = (A + Bn)(\frac{1}{2})^n \]

Now suppose that a particular solution is of the form \[ x_n^p = an+b. \] Then

\[ x_{n+1}^p = a(n+1)+b \]

\[ x_{n+2}^p = a(n+2)+b \]

So \((*)\) becomes

\[ 4[an+2+b] + 4[a(n+1)+b] + an+b = 3n \]

\[ 9a\cdot n + (9a + 9b) = 3n + 0 \]

So \[ 9a = 3 \quad \text{(coeff. of } n) \]

and \[ 12a + 9b = 0 \quad \text{(const. term)} \]

\[ a = \frac{1}{3} \quad \text{and} \quad b = -12a = -\frac{12}{3} = -\frac{4}{1} = -\frac{4}{3} \]

A quick check shows that \(\frac{1}{3}n - \frac{4}{3}\) is indeed a solution of \((*)\).

So the general solution of \((*)\) is

\[ x_n = x_n^c + x_n^p = (A + Bn)(\frac{1}{2})^n + \frac{1}{3}n - \frac{4}{3} \]

3. We have \[ x_{n+2} + x_{n+1} - 6x_n = 4 \quad \ldots \quad (*) \]

Homog. eq is: \[ x_{n+2} + x_{n+1} - 6x_n = 0 \]

Aux eq is: \[ E^2 + E - 6 = 0 \]

\[ (E+3)(E-2) = 0 \]

So \[ E = 2 \] or \(-3\)

\[ x_n^c = A \cdot 2^n + B \cdot (-3)^n \]
3. Suppose that a particular solution is of the form $x_n^p = c$. Then $x_{n+1}^p = c$, $x_{n+2}^p = c$.

So (3) becomes

$$c + c - 6c = 4$$

$\therefore -4c = 4$ and so $c = -1$.

A quick check shows that $x_n = -1$ is indeed a solution of (3). Thus the general solution is

$$x_n = x_n^c + x_n^p = A \cdot 2^n + B \cdot (-3)^n - 1.$$ 

But $x_0 = 3$ and $x_1 = 2$, so

1. $$A + B - 1 = 3$$
2. $$2A - 3B - 1 = 2$$
3. $$(1) \times 3: \quad 5A + 3B - 3 = 9$$
4. $$(2) + (3) \Rightarrow 5A - 4 = 11$$

$\therefore A = 3$ and $B = 3 - A + 1 = 3 - 3 + 1 = 1$.

So the solution is $x_n = 3 \cdot 2^n + (-3)^n - 1$.

4. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

Then $-2f(x) = -2a_0 x - 2a_1 x^2 - \ldots - 2a_{n-1} x^n - \ldots$

Also $\frac{1}{1-x} = 1 + 1 \cdot x + 1 \cdot x^2 + \ldots + 1 \cdot x^n + \ldots$

: $(1-2x) f(x) + \frac{1}{1-x} = a_0 + 1 + (a_1 - 2a_0 + 1) x + (a_2 - 2a_1 + 1) x^2 + \ldots + (a_{n-2} - 2a_{n-1} + 1) x^n + \ldots$

$= a_0 + 1$, since $a_n - 2a_{n-1} + 1 = 0$, $n \geq 1$

$= 1$, since $a_0 = 0$.

: $(1-2x) f(x) = 1 + \frac{-1}{1-x} = \frac{1}{1-x} = \frac{-x}{1-x}$.
4. \[ f(x) = \frac{-x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x} \]

So \[-x = A(1-x) + B(1-2x)\]

Putting \(x = \frac{1}{2}\) gives \[-\frac{1}{2} = A(1-\frac{1}{2}) + B \cdot 0\]

\[-\frac{1}{2} = \frac{1}{2}A \quad \therefore \quad A = -1\]

Putting \(x = 1\) gives \[-1 = A \cdot 0 + B(1-2)\]

\[-1 = -B \quad \therefore \quad B = 1\]

So \[ f(x) = \frac{-1}{1-2x} + \frac{1}{1-x} \]

\[= -(1 + 2x + (2x)^2 + \ldots + (2x)^n + \ldots) + (1 + x + x^2 + \ldots + x^n + \ldots) \]

So \[a_n = \text{coefficient of } x^n \text{ in the expansion of } f(x)\]

\[= -2^n + 1\]

5. Since \(G\) is a connected planar graph, we know that \(f = e+2-n\) (Euler's formula)

(a) Since \(G\) has no cycles of length 3, 4 or 5, we see that, in any planar representation of \(G\), each face is bounded by at least 6 edges.

\[\therefore (\text{No. of faces}) \times 6 \leq \text{No. of edges counted from faces}\]

\[\therefore 6f \leq 2e \quad \text{because each edge is counted in at most 2 faces}.\]

\[\therefore 3f \leq e.\]

(b) Substituting for \(f\) by using Euler's formula gives:

\[3(e+2-n) \leq e\]

\[\therefore 3e - e \leq 3n - 6. \quad \text{Thus } 2e \leq 3n - 6.\]
6. There are two cases:

Case (i): $G$ has no vertex of degree 0.
In this case the maximum possible degree is $n-1$ (because a vertex can only be adjacent to the other $n-1$ vertices at best). So the set of possible degrees is $\{1, 2, 3, \ldots, n-1\}$.

Since we have $n$ vertices, it follows from the Pigeon Hole Principle that two vertices must have the same degree.

Case (ii): $G$ has a vertex of degree 0.
In this case a vertex can only be adjacent to at most $n-2$ other vertices (because it and the vertex of degree 0 are excluded). So the maximum possible degree is $n-2$, and hence the set of possible degrees is $\{0, 1, 2, \ldots, n-2\}$.

We again have $n-1$ choices and $n$ vertices. By the pigeon Hole Principle it follows that two vertices must have the same degree.