Chapter 4 - The Inclusion-Exclusion Principle

§1. Two forms of the Inclusion-Exclusion Principle

Exercise 1

How many integers in the set \( U = \{1, 2, 3, \ldots, 1000\} \) are divisible by 3, 5, or 7.

Solution

Let \( A = \{n \in U : n \text{ is divisible by } 3\} \),
\( B = \{n \in U : n \text{ is divisible by } 5\} \),
and \( C = \{n \in U : n \text{ is divisible by } 7\} \).

Then the answer to our problem is
\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]

Now, \( A = \{3(1), 3(2), 3(3), \ldots, 3(k)\} \) where \( k \) is the largest integer \( \leq 1000/3 \). So,
\[|A| = \left\lfloor \frac{1000}{3} \right\rfloor.
\]
Similarly, \( |B| = \left\lfloor \frac{1000}{5} \right\rfloor \) and \( |C| = \left\lfloor \frac{1000}{7} \right\rfloor \).

Also, \( A \cap B = \{n \in U : n \text{ is divisible by both } 3 \text{ & } 5\} \)
\[= \{n \in U : n \text{ is divisible by lcm}(3, 5)\}
\[= \{n \in U : n \text{ is divisible by } 15\}.
\]
So,
\[|A \cap B| = \left\lfloor \frac{1000}{15} \right\rfloor.
\]
Similarly, \( |A \cap C| = \left\lfloor \frac{1000}{3} \right\rfloor \) and \( |B \cap C| = \left\lfloor \frac{1000}{35} \right\rfloor \).

And, \( |A \cap B \cap C| = \left\lfloor \frac{1000}{105} \right\rfloor \). Thus,
\[
|A \cup B \cup C| = \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{1000}{35} \right\rfloor
\]
\[+ \left\lfloor \frac{1000}{105} \right\rfloor = 333 + 200 + 142 - 66 - 47 - 28 + 9 = 543.
\]
Definition: Let $A_1, A_2, \ldots, A_n$ be subsets of a universal set $U$. A positive set w.r.t. $U$ & $A_1, A_2, \ldots, A_n$ is any set of the form $\bigcup A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}$ where $\langle i_1, i_2, \ldots, i_k \rangle$ is any subsequence (including the empty subsequence $\langle \rangle$) of the sequence $\langle 1, 2, 3, \ldots, n \rangle$.

We usually leave out the $U$ when $\langle i_1, \ldots, i_k \rangle$ is not the empty sequence — and we also leave out the intersection signs. So we will write $\bigcup A_2 \cap A_4 \cap A_5$ as $A_2 A_4 A_5$.

Note: Recall that a sequence is just a function with domain $\langle 1, 2, 3, \ldots, n \rangle$. The subsequences of $f$ are obtained by restricting $f$ to the different subsets of $\langle 1, 2, 3, \ldots, n \rangle$. So from the sequence $\langle f(1), f(2), \ldots, f(m) \rangle$ we can get $\langle \rangle$ by restricting $f$ to $\emptyset$, $\langle f(2), f(3) \rangle$ by restricting $f$ to $\{2, 3\}$, $\langle f(1), f(3), f(4) \rangle$ by restricting $f$ to $\{1, 3, 4\}$. Since there are $2^n$ subsets of $\langle 1, 2, 3, \ldots, n \rangle$, there will be $2^n$ different subsequences of $\langle f(1), f(2), \ldots, f(m) \rangle$. This immediately tells us that there will be $2^n$ positive sets because there are $2^n$ subsequences $\langle i_1, \ldots, i_k \rangle$ of $\langle 1, 2, 3, \ldots, n \rangle$.

Let us analyze the positive sets in more details.
Def. The order of a positive set is the number of \( A_i \)'s it contains. In other words, it is the length of the subsequence \( \langle i_1, \ldots, i_k \rangle \) from which it came.

Prop. 1 (a) There are \( \binom{n}{k} \) positive sets of order \( k \).
(b) Consequently, there are (again) \( 2^n \) positive sets.

Proof (a) The number of positive sets of order \( k \) is the number of subsequences \( \langle i_1, \ldots, i_k \rangle \) of \( \langle 1, 2, 3, \ldots, n \rangle \) with \( k \) terms. But this is just the number of \( k \)-subsets of \( \{1, 2, \ldots, n\} \) because \( \langle i_1, \ldots, i_k \rangle \) has to be in increasing order. Since there are \( \binom{n}{k} \) \( k \)-subsets of \( \{1, 2, \ldots, n\} \), there will be \( \binom{n}{k} \) positive sets of order \( k \).

(b) The number of positive sets is equal to
\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n \] by a previous result.

Positive sets of Order 0: \( \emptyset \). (3)

Order 1: \( A_1, A_2, \ldots, A_n \). (4)

Order 2: \( A_1A_2, A_1A_3, \ldots, A_1A_n, A_2A_3, \ldots, A_{n-1}A_n \).

Order \( n \): \( A_1A_2A_3 \ldots A_n \). (5)

Order or (in more details): \( A_1A_2, A_1A_3, A_1A_4, \ldots, A_1A_n \), \( A_2A_3 \), \( A_2A_4, \ldots, A_2A_n \), \( A_3A_4, A_3A_5, \ldots, A_3A_n \), \( A_4A_5, A_4A_6, \ldots, A_4A_n \), \( \ldots, A_{n-2}A_{n-1}, A_{n-2}A_n, A_{n-1}A_n \).
So the number of positive sets of order 2 will be \((n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{n(n-1)}{2}\) as indicated before.

Theorem 2 (Inclusion-Exclusion Theorem - First version)
Let \(U\) be a universal set and \(A_i = \{x \in U: x\) has property \(P_i\}\) for \(i = 1, 2, \ldots, n\).
Then the number of elements of \(U\) with none of the properties \(P_i\) is given by

\[
\left| A_1^c A_2^c A_3^c \ldots A_n^c \right| = \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} A_{i_2} \ldots A_{i_k}| \right\}
\]

\[
= |U| - \sum_{i=1}^{n} |A_i| + \sum_{1 \leq i < j \leq n} |A_i A_j| - \ldots + (-1)^n \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} \ldots A_{i_k}| + \ldots + (-1)^n |A_1 A_2 \ldots A_n|.
\]

Proof: We shall prove (x) by showing that an element of \(U\) with none of the properties \(P_1, \ldots, P_n\) is counted once in the RHS (x) and that an element with at least one of the properties \(P_1, \ldots, P_n\) is counted zero times in the RHS (x).

Now if \(x\) has none of the properties \(P_1, \ldots, P_n\), then \(x\) will be counted exactly once in \(|U|\) and zero times in all the other terms of the RHS (x).

Also if \(x\) has the properties \(P_{i_1}, P_{i_2}, \ldots, P_{i_k}\)
then \( x \) will be counted in the RHS(*)

\[
\sum_{i=0}^{k} \binom{k}{i} = \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^i \binom{k}{i} = 0 \text{ times.}
\]

\( x \) is counted in U \( \Rightarrow \) No. of times
\( x \) is counted in the sets of order 1 \( \Rightarrow \) No. of times
\( x \) is counted in the sets of order 2 \( \Rightarrow \) No. of times
\( x \) is counted in the sets of order k \( \Rightarrow \) No. of times

\[\therefore \text{ LHS}(*) = \text{ RHS}(*) \text{, So the result follows.} \]

**Corollary 3 (Inclusion-Exclusion Theorem - Version 2)**

Let \( A_1, A_2, \ldots, A_n \) be as in Theorem 2. Then

\[
|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}
\]

\[
= \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i A_j| + \cdots + (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} \cdots A_{i_k}|
\]

\[+ \cdots + (-1)^{n-1} \cdot |A_1 A_2 \cdots A_n|.
\]

**Proof:** We know that

\[
|A_1 \cup A_2 \cup \ldots \cup A_n| = |U| - |(A_1 \cup A_2 \cup \ldots \cup A_n)^c|
\]

\[
= |U| - |A_1^c \cap A_2^c \cap \cdots \cap A_n^c|
\]

\[
= |U| - |A_1^c A_2^c A_3^c \cdots A_n^c|
\]

\[
= |U| - \text{ RHS(*) of Theorem 2}
\]

\[
= |U| - \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\}
\]

\[
= \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i A_j| + \cdots + (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} \cdots A_{i_k}|
\]

\[+ \cdots + (-1)^{n-1} \cdot |A_1 A_2 \cdots A_n|.
\]
§2. Two forbidden position problems

Recall that a permutation of \( \{1, 2, 3, \ldots, n\} \) is a sequence of \( n \) distinct elements of \( \{1, 2, 3, \ldots, n\} \). In other words, a permutation of \( \{1, 2, 3, \ldots, n\} \) is a bijection (one-to-one correspondence) from \( \{1, 2, 3, \ldots, n\} \) to itself.

So the permutation \( \langle 2, 3, 1 \rangle \) is really the bijection \( (\begin{array}{ccc} 1 & z & 3 \\ 2 & z & 3 \end{array}) \), i.e., the function which sends 1 to 2, 2 to 3, and 3 to 1.

Def. A derangement of \( \{1, 2, \ldots, n\} \) is a permutation of \( \{1, 2, \ldots, n\} \) in which no element is in its natural position, i.e., in which no element goes to itself.

Ex.1 \( \langle 2, 3, 1 \rangle \) & \( \langle 3, 1, 2 \rangle \) are the derangements of \( \{1, 2, 3\} \).
\( \langle 2, 1, 3 \rangle \), \( \langle 1, 3, 2 \rangle \), \( \langle 3, 2, 1 \rangle \) & \( \langle 1, 2, 3 \rangle \) are not derangements of \( \{1, 2, 3\} \).

Ex.2 Let \( D_n \) = set of all derangements of \( \{1, 2, \ldots, n\} \) and \( D_n = |D_n| \). Then:
\[ D_0 = \{\langle \rangle\} \] so \( D_0 = 1 \)
\[ D_1 = \emptyset \] \( D_1 = 0 \)
\[ D_2 = \{\langle 2, 1 \rangle\} \] \( D_2 = 1 \)
\[ D_3 = \{\langle 2, 3, 1 \rangle, \langle 3, 1, 2 \rangle\} \) \( D_3 = 2 \).

We will later see that \( D_4 = 9 \).
Theorem 4
The number of derangements of \{1, 2, \ldots, n\} is given by
\[ D_n = n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right\} \]
\[ = n! \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right\} \]

Proof: Let \( U \) = set of all permutations of \{1, 2, \ldots, n\}
Put \( A_i \) = set of all permutations in \( U \)
with \( i \) going to itself. \( i = 1, \ldots, n \).
Then \[ |U| = n! \]
\[ |A_i| = (n-1)! \]
\[ \left( \begin{array}{c} 1, 2, \ldots, i, \ldots, n \end{array} \right) \]
Also \( A_i A_j = A_i \cap A_j \)
\[ = \text{set of all permutations in } U \]
with \( i \) going to \( i \) \& \( j \) going to \( j \). So
\[ |A_i A_j| = (n-2)! \]
for \( 1 \leq i < j \leq n \).
In general \[ |A_{i_1} A_{i_2} \cdots A_{i_k}| = (n-k)! \]
(for \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \)) for each of the \( \binom{n}{k} \) positive
sets of order \( k \). So by the Inclusion-Exclusion Theorem we get
\[ D_n = |A_1^c \cap A_2^c \cap \ldots \cap A_n^c| = |A_1^c A_2^c \ldots A_n^c| \]
\[ = \sum_{k=0}^{n} (-1)^k \left\{ \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} A_{i_2} \cdots A_{i_k}| \right\} \]
\[ = \sum_{k=0}^{n} (-1)^k \cdot \left\{ \binom{n}{k} \cdot (n-k)! \right\} \]
\[ = \sum_{k=0}^{n} (-1)^k \cdot \left\{ \frac{n!}{k!} \cdot (n-k)! \right\} = n! \cdot \sum_{k=0}^{n} (-1)^k \frac{(n-k)!}{k!} \]
\[ = n! \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right\}. \]
Prop 5
(a) For any \( n \geq 1 \), \( D_n = \{n, \Delta n-1\} + (-1)^n \)
(b) For any \( n \geq 2 \), \( D_n = (n-1) \cdot (\Delta n-1 + \Delta n-2) \)

Proof:
(a) \( D_n = n! \left[ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!} + \frac{1}{n!} \right] \)

\[ = n \cdot (n-1)! \left[ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^{n-1} \frac{1}{(n-1)!} \right] + n! \cdot \frac{(-1)^n}{n!} \]

\[ = n \cdot \Delta n-1 + (-1)^n. \]

(b) \( D_n = \{n, \Delta n-1\} + (-1)^n \)

\[ = (n-1) \cdot D n-1 + \Delta n-1 + (-1)^n \]

\[ = (n-1) \cdot D n-1 + (n-1) \cdot \Delta n-2 + (-1)^{n-1} \cdot \Delta n-1 \]

\[ = (n-1) \cdot [\Delta n-1 + \Delta n-2] . \]

Ex. 3
We have already seen that \( \Delta_3 = 2 \). So

\[ D_4 = 4 \cdot \Delta_3 + (-1)^4 = 4 \cdot 2 + 1 = 9 \]

\[ D_5 = 5 \cdot D_4 + (-1)^5 = 5 \cdot 9 + (-1) = 44 \]

\[ D_6 = 6 \cdot D_5 + (-1)^6 = 5 \cdot (44) + 1 = 265 \]

\[ D_7 = 7 \cdot D_6 + (-1)^7 = 7 \cdot (265) + (-1) = 1854 . \]

Ex. 4
In how many ways can we return the watches of 3 men and 3 ladies so that
(a) no person gets their own watch
(b) no person gets their own watch and each lady receives a ladies watch.

Sol. (a) Answer = \( D_6 = 265 \) ways

(b) Answer = \( (\Delta_3) (\Delta_3) = 2(2) = 4 \) ways, but each lady will get a ladies watch & the men will get men's watches.
Ex. 5  In how many ways can we return the cars of 5 super-models so that
(a) no super-model gets her own car
(b) exactly one super-model gets her own car
(c) exactly two super-models get her own car
(d) at most two super-models get her own car
(e) at least two super-models get her own car

(a) Answer = D_5 = 44

(b) There are \( \binom{5}{1} \) ways to choose the one super-model who will get her own car. Then we derange the cars of the other 4 super-models in D_4 ways. So our answer = \( \binom{5}{1} \cdot 4! = 45 \).

(c) There \( \binom{5}{2} \) ways to choose the two super-models who will get their own cars. Then we derange the cars of the other 3 super-models in D_3 ways. Answer will be \( \binom{5}{2} \cdot D_3 = \frac{5!}{2!} \cdot 3! = 20 \).

(d) Answer = Ans(a) + Ans(b) + Ans(c)

\[
= 44 + 45 + 20 = 109
\]

(e) Answer = total no. of ways of permuting the cars - the answers in (a) & (b)

\[
= 5! - [\text{Ans(a) + Ans(b)}] \\
= 120 - (44 + 45) = 31.
\]

(e') We can also add the number of ways 2, 3, 4 & 5 super-models get their own cars to get

\[
\text{Ans(e)} = \binom{5}{2} \cdot D_3 + \binom{5}{3} \cdot D_2 + \binom{5}{4} \cdot D_1 + \binom{5}{5} \cdot D_0
\]

\[
= 10(2) + 10(1) + 5(0) + 1(1)
\]

\[
= 20 + 10 + 1 = 31.
\]
Def. A non-consecutive permutation of \( \{1, 2, \ldots, n\} \) is a permutation of \( \{1, 2, \ldots, n\} \) in which there is no pair of consecutive terms of the form \( \langle i, i+1 \rangle \). In other words, if we view the permutation as a bijection \( f \), then there is no value of \( j \) such that \( f(j+1) = f(j)+1 \), for \( j = 1, 2, \ldots, n-1 \).

Ex. 6 (a) \( \langle 1, 3, 2 \rangle \), \( \langle 2, 1, 3 \rangle \), and \( \langle 3, 2, 1 \rangle \) are non-consecutive permutations of \( \{1, 2, 3\} \).
(b) \( \langle 1, 2, 3 \rangle \), \( \langle 2, 3, 1 \rangle \), and \( \langle 3, 1, 2 \rangle \) are not non-consecutive permutations of \( \{1, 2, 3\} \).

Notation: Let \( \mathcal{Q}_n \) = set of all non-consecutive permutations of \( \{1, 2, \ldots, n\} \) and \( Q_n = |\mathcal{Q}_n| \). Then
\[
\begin{align*}
\mathcal{Q}_0 &= \{\} \quad \text{so} \quad Q_0 = 1 \\
\mathcal{Q}_1 &= \{\langle 1 \rangle \} \quad \text{so} \quad Q_1 = 1 \\
\mathcal{Q}_2 &= \{\langle 2, 1 \rangle \} \quad \text{so} \quad Q_2 = 1 \\
\mathcal{Q}_3 &= \{\langle 1, 3, 2 \rangle, \langle 2, 1, 3 \rangle, \langle 3, 2, 1 \rangle\} \quad \text{and} \quad Q_3 = 3 \\
\text{Later on we will see that} \quad Q_4 = 12.
\end{align*}
\]

Theorem: The number of non-consecutive permutations of \( \{1, 2, \ldots, n\} \) is
\[
Q_n = \sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot (n-k)!
\]

Proof: Let \( U = \) set of all permutations of \( \{1, 2, \ldots, n\} \) and \( A_i = \) set of all permutations in \( U \) which contain \( \langle i, i+1 \rangle \) as consecutive terms.
Then $A_1$ = set of permutations in $U$ with $\langle 1,2 \rangle$ as a pair of consecutive terms
= set of permutations of $\{12,3,4,\ldots,n\}$
So $|A_1| = (n-1)!$. Similarly, $|A_i| = (n-1)!$
for $i = 2, \ldots, n-1$ as well.

Also $A_1 A_2$ = set of permutations in $U$ with both $\langle 1,2 \rangle$ & $\langle 2,3 \rangle$ as pairs of consecutive terms
= set of permutations of $\{123,4,\ldots,n\}$
So $|A_1 A_2| = (n-2)!$

And $A_1 A_3$ = set of permutations in $U$ with both $\langle 1,2 \rangle$ & $\langle 3,4 \rangle$ as pairs of consecutive terms
= set of permutations of $\{1234,5,\ldots,n\}$
So $|A_1 A_3| = (n-2)!$

From this we can see that for any $i$ & $j$ with $1 \leq i < j \leq n-1$, we have $|A_i A_j| = (n-2)!$

In general we can also see that for any $\langle i_1, \ldots, i_k \rangle$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$, we have $|A_{i_1} A_{i_2} \cdots A_{i_k}| = (n-k)!$
So $Q_n$ = set of all permutations in $U$ with no pair of consecutive terms

$= |A_1 A_2 \cdots A_{n-1}| = \sum_{k=0}^{n-1} (-1)^k \sum_{1 \leq i_1 \ldots \leq i_k \leq n-1} |A_{i_1} \cdots A_{i_k}|$
$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!$
$= (n-1)_n \cdot n! - (n-1)_1 \cdot (n-1)! + \binom{n-2}{2} \cdot (n-2)! - \cdots + (-1)^{n-1} \binom{n-1}{1} n!$
Prop. 7 For any \( n \geq 1 \), \( Q_n = D_n + D_{n-1} \).

Proof: 
\[
Q_n = \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} (n-k)!
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!(n-1-k)!} \frac{1}{k!} (n-k)
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!} \frac{1}{(n-1-k)!} (n-k) \frac{k}{k!}
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{n!}{k!} \left[ \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{(k+1)!} \right]
\]

\[
= \sum_{k=0}^{n-1} (-1)^k \frac{n!}{k!} \left[ \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{(k+1)!} \right] - \frac{n!}{k!} \sum_{k=0}^{n-2} (-1)^k \frac{(n-1)!}{(k+1)!}
\]

\[
= n! \left[ \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right] = (-1)^n + \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)!}{k!} = (-1)^n \frac{n!}{(n-1)!}
\]

\[
= n! \left[ \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right] + (n-1)! \left[ \sum_{k=0}^{n-1} (-1)^k \frac{1}{k!} \right] = (-1)^n \left[ (-1)^{n-1} + 1 \right]
\]

\[
= D_n + D_{n-1}.
\]

\( \square \)

Ex. 6 Five sisters walk to school in a straight line. In how many ways can they walk back home in a straight line so that no sister sees the same person in front of them again.

Sol. Answer = \( Q_5 = 25 + 24 = 44 + 9 = 53 \) ways.
§3. Solutions of $x_1 + \cdots + x_n = r$ with constraints and $r$-combinations of finite multi-sets

Ex. 1. How many integer-solutions of the equation $x_1 + x_2 + x_3 = 17$ are there with $x_1 \geq 3$, $x_2 \geq 5$, and $x_3 \geq 2$?

Sol. Let $X_1 = Y_1 + 3$, $X_2 = Y_2 + 5$, and $X_3 = Y_3 + 2$. Then our answer will be the same as the number of integer-solutions of the equation $(Y_1 + 3) + (Y_2 + 5) + (Y_3 + 2) = 17$ with $Y_1 + 3 \geq 3$, $Y_2 + 5 \geq 5$, and with $Y_3 + 2 \geq 2$.

This is the same as the number of integer-solutions of $Y_1 + Y_2 + Y_3 = 7$ with $Y_1 \geq 0, Y_2 \geq 0, Y_3 \geq 0$. And we know that this is the same as the number of ways of arranging 8 1's, 5 2's, and 5 in a row, i.e., $9 \choose 2 = \frac{(7+2)!}{2!} = \frac{9!}{(7-3)!2!} = \frac{9 \times 8}{2 \times 1} = 36$.

So our final answer is $\binom{9}{2} = \frac{9 \times 8}{2 \times 1} = 36$.

Ex. 2. Let $M = \{\infty, a, \infty, b, \infty, c\}$. How many 20-combinations of $M$ are there with $\geq 3$ a's, $\geq 5$ b's, and $\geq 2$ c's?

Sol. Let $X_1 = \text{no. of a's in a 20-combination of $M$}$, $X_2 = \text{no. of b's in the same 20-comb. of $M$}$ and $X_3 = \text{no. of c's in the same 20-comb. of $M$}$. Then our answer to the problem will be the number of integer-solutions of the equation.
\[ x_1 + x_2 + x_3 = 20 \text{ with } x_1 \geq 3, \ x_2 \geq 5, \text{ and } x_3 \geq 2. \]

And from example 1, we found that this is 36.

**Sol. 2**

Now there is another way to do this problem.

Let \( A \) = set of all 7-comb. of \( M \) and
\[ A' = \text{ set of each 7-comb. in } A \text{ plus } [3a, 5b, 2c]. \]

Then, \( A' \) is a 17-comb. of \( M \) because each element of \( A' \) was obtained by adding a multi-set with 10 elements to a 7-comb. of \( M \). Also each element of \( A' \) is a 17-comb. of \( M \) with \( \geq 3 \) a's, \( \geq 5 \) b's, and \( \geq 2 \) c's.

Since there is an obvious bijection from \( A \) to \( A' \), it follows that
\[ |A'| = \text{ No. of 17-comb. of } M \text{ with } \geq 3 \text{ a's, } \geq 5 \text{ b's, } \geq 2 \text{ c's}. \]
\[ = \text{ No of 7-comb. of } M = |A| = \binom{7+3-1}{3-1} = \binom{9}{2}. \]

So, our final answer is \( \binom{9}{2} = \frac{9!}{2!} = 36 \) again.

**Ex. 3**

How many 15-combinations of the finite multi-set \( F = [4a, 6b, 20c] \) are there?

**Sol.**

Let \( M = [\infty a, \infty b, \infty c] \) and put
\[ U = \text{ set of all 15-combinations of } M \]
\[ A = \text{ set of all 15-comb. in } U \text{ with } > 4 \text{ a's}, \]
\[ B = \text{ set of all 15-comb. in } U \text{ with } > 6 \text{ b's}, \]
\[ \text{ & } C = \text{ set of all 15-comb. in } U \text{ with } > 20 \text{ c's}. \]

Then
\[ A = \text{ set of all 10-comb. of } M \text{ with 5 extra a's added}, \]
\[ B = \text{ set of all 8-comb. of } M \text{ with 7 extra b's added}, \]
\[ \text{ & } C = \emptyset, \text{ bec. a 15-comb. cannot have } \geq 21 \text{ c's.} \]
$U = \binom{15+3-1}{3-1}, \quad |A| = \binom{10+3-1}{3-1}, \quad B = \binom{8+3-1}{3-1}$

Also $A \cap B = \text{set of all 15-comb. of } M \text{ with at least 5 a's and 7 b's}$

$= \text{set of all 3-comb. of } M \text{ with } [5a, 7b] \text{ added}$

$\Rightarrow |A \cap B| = \binom{3+3-1}{3-1}$. Since we want $\leq 4$ a's, $\leq 6$ b's, and $\leq 20$ c's in our 15-comb. of $M$, our final answer would be $|A^c \cap B^c \cap C^c|$

But by the Inclusion-Exclusion Theorem:

$|A^c \cap B^c \cap C^c| = U - |A| - |B| - |C| + |A \cap B| + |B \cap C|$

$- |A \cap B \cap C|$

$= \binom{17}{2} - \binom{12}{2} - \binom{10}{2} - 0 + \binom{5}{2} + 0 + 0 - 0$

$= \binom{17}{2} + \binom{5}{2} - \binom{12}{2} - \binom{10}{2}$

because $C$, $A \cap C$, $B \cap C$, and $A \cap B \cap C$ are all empty.

$= \frac{17 \cdot 16}{2} + \frac{5 \cdot 4}{2} - \frac{12 \cdot 11}{2} - \frac{10 \cdot 9}{2} = 136 + 10 - 66 - 45 = 35.$

Ex. 4 How many 26-comb. of the finite multi-set $F = [4, a, 6, b, 20, c]$: are there?

Sol. 1 Again let $M = [ \infty, a, \infty, b, \infty, c ]$ and put

$U = \text{set of all 26-comb. of } M$

$A = \text{set of all 26-comb. in } U \text{ with } \geq 4 \text{ a's (75a's)}$

$B = \text{set of all 26-comb. in } U \text{ with } \geq 6 \text{ b's (776's)}$

$C = \text{set of all 26-comb. in } U \text{ with } \geq 20 \text{ c's (7216's)}$

Then $A \cap B \cap C$

$A \cap B = \text{set of all 21-comb. of } M \text{ with } [5a] \text{ added to each 21-comb.}$

$B \cap C = \text{set of all 19-comb. of } M \text{ with } [7b] \text{ added to each 19-comb.}$

$C \cap A = \text{set of all 5-comb. of } M \text{ with } [21c] \text{ added to each 5-comb.}$

$A \cap B \cap C = \text{set of all 14-comb. of } M \text{ with } [5a, 7b] \text{ added to each 14-comb.}$
\[ A \cap C = \text{set of all 0-comb. of } M \text{ with } [50, 21C] \text{ added to each 0-comb.} \]
\[ B \cap C = \emptyset \text{ and } A \cap B \cap C = \emptyset. \text{ So} \]

Number of 26-comb. of \( F = |A^c \cap B^c \cap C^c| \)
\[ = |U| - |A^c| - |B^c| - |C^c| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \]
\[ = \left( \binom{26}{3} \right) - \left( \binom{21}{3} \right) - \left( \binom{19}{3} \right) - \left( \binom{5}{3} \right) + \left( \binom{14}{3} \right) + \left( \binom{3}{3} \right) - 15 \]
\[ = \left( \binom{28}{2} \right) + \left( \binom{16}{2} + \binom{2}{2} \right) - \left( \binom{23}{2} - \binom{21}{2} \right) - \left( \binom{7}{2} \right) = 499 - 484 = 15. \]

**Sol. 2**

But there is a much quicker way to do the same problem. We just have to observe that No. of 26-comb. of \( F = \) No. of 4-comb. of \( F \) because \( F \) has 30 elements. If we want to pick 26 elements out of \( F \), we can just pick 4 elements to leave behind and get the same answer. So let \( M = \{000, 001, 011, 101, 002, 022, 202, 222, 003, 033, 303, 013, 103, 313, 223, 323, 333, 024, 034, 124, 134, 214, 234, 314, 324, 344\} \)

& \( U = \text{set of all 4-comb. of } M. \text{ Put} \)

\( A = \text{set of all 4-comb. in } U \text{ with } > 4 \text{ a's} \)
\( B = \text{set of all 4-comb. in } U \text{ with } > 6 \text{ b's} \)
\( C = \text{set of all 4-comb. in } U \text{ with } > 20 \text{ c's} \)

Then \( A = B = C = \emptyset \) and \( A \cap B = A \cap C = B \cap C = A \cap B \cap C = \emptyset \) also. So

Number of 4-comb. of \( F = |A^c \cap B^c \cap C^c| \)
\[ = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C| \]
\[ = \left( \binom{4}{3} \right) = \left( \binom{6}{2} \right) = 15 \text{ (as before)} \]

So as you can see, it pays to be a little smart and think a little bit before trying to solve the problem. By the way, this trick would not have worked with Ex. 3.
**Def.** Let $U$ be a universal set and $A_1, \ldots, A_n$ be subsets of $U$. An ultimate set with respect to $A_1, \ldots, A_n$ is any set of the form $X_1 \cap X_2 \cap \ldots \cap X_n$ where $X_i = A_i$ or $A_i^c$ for $i = 1, \ldots, n$.

**Prop. 8.** There are $2^n$ ultimate sets w.r.t. $A_1, \ldots, A_n$.

**Proof.** For each $X_i$ we have 2 choices. Since there are $n$ $X_i$'s we will get $2^n$ choices & so $2^n$ ultimate sets.

**Ex. 5.** Let $U$ be a universal and $A_1 \& A_2$ be subsets of $U$. Find all the ultimate sets w.r.t. $A_1 \& A_2$.

**Sol.** They are $A_1 \cap A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2$, $A_1^c \cap A_2^c$.

**Prop. 9.** The ultimate sets w.r.t. $A_1, \ldots, A_n$ are all pairwise disjoint.

**Proof.** Suppose $Z_1$ and $Z_2$ are ultimate sets. Let $Z_1 = X_1 \cap X_2 \cap \ldots \cap X_n$ and $Z_2 = Y_1 \cap Y_2 \cap \ldots \cap Y_n$ where $X_i = A_i$ or $A_i^c$ and $Y_i = A_i$ or $A_i^c$.

Then for some $i_0$, $X_{i_0}$ & $Y_{i_0}$ must be different (because if $X_i = Y_i$ for each $i$, then $Z_1$ & $Z_2$ would be the same). Hence

$Z_1 \cap Z_2 = (X_1 \cap X_2 \cap \ldots \cap X_n) \cap (Y_1 \cap Y_2 \cap \ldots \cap Y_n)$

$\subseteq X_{i_0} \cap Y_{i_0} = \emptyset$ bec. $X_{i_0} \cap Y_{i_0} = A_{i_0} \cap A_{i_0}^c$.

i.e. $Z_1 \cap Z_2 = \emptyset$. Hence any two ultimate sets are disjoint.
Consistency of Data

Ex. 6. Suppose we are told: among the Math majors:
(a) 26 of them are taking Graph Theory, or Combinatorics, or both;
(b) 9 of them are taking Graph Theory;
(c) 8 of them are taking Combinatorics but not Graph Theory; and
(d) 15 of them are not taking Combinatorics.

Let \( U \) = set of Math majors, \( A \) = set of Math majors taking Graph Theory, and \( B \) = set of math majors taking Combinatorics. Put
\[
X_1 = |A\cap B|, \quad X_2 = |A\cap \overline{B}|, \quad X_3 = |\overline{A}\cap B| \quad \text{and} \quad X_4 = |\overline{A}\cap \overline{B}|
\]
Then the data translates to the system of equations
\[
\begin{align*}
X_1 + X_2 + X_3 &= 20 \\
X_1 + X_2 &= 9 \\
X_3 &= 8 \\
X_3 + X_4 &= 15
\end{align*}
\]

Then there will be 3 possibilities:

I: The system has no solution: In this case, the data will be inconsistent.

IIA: The system has a unique solution: In this case, the data is consistent & it determines the situation.

IIB: The system has more than one solution: In this case, the data is consistent but it does not determine the situation.

In Ex. 6, the system has no solution. So the data is inconsistent.