ODE

Philippe Rukimbira

Department of Mathematics
Florida International University
4.4 The method of Variation of parameters

1. Second order differential equations (Normalized, standard form!).

\[ y'' + P(x)y' + Q(x)y = f(x) \]

Suppose \( y_1 \) and \( y_2 \) form a fundamental set of solutions on an interval \( I \) for

\[ y'' + P(x)y' + Q(x)y = 0 \]

We seek functions \( u_1(x) \) and \( u_2(x) \) such that:

\[ y_p = u_1 y_1 + u_2 y_2 \]

is a particular solution of the nonhomogeneous differential equation.
In that case,

\[ y'_p = u_1 y'_1 + y_1 u'_1 + u_2 y'_2 + y_2 u'_2 \]

We can impose the additional condition on \( u_1 \) and \( u_2 \):

\[ y_1 u'_1 + y_2 u'_2 = 0 \]

That is equivalent to

\[ y'_p = u_1 y'_1 + u_2 y'_2 \]
From there,

\[ y''_p = u'_1 y'_1 + u_1 y''' + u'_2 y'_2 + u_2 y''_2 \]

Now, substitute for \( y_p, y'_p \) and \( y''_p \) into the nonhomogeneous differential equation"

\[ y''_p + P y'_p + Q y_p = f(x) \]

which becomes:

\[ u'_1 y'_1 + u_1 y''' + u'_2 y'_2 + u_2 y''_2 + P u_1 y'_1 + P u_2 y'_2 + Q u_1 y_1 + Q u_2 y_2 = f(x) \]
Reorganizing leads to

\[ u_1' y_1' + u_1(y_1'' + Py_1' + Qy_1) + u_2' y_2' + u_2(y_1'' + Py_2' + Qy_2) = f(x). \]

and finally, one obtains a second condition on \( u_1 \) and \( u_2 \)

\[ y_1' u_1' + y_2' u_2' = f(x) \]
What we have now is a system of two equations involving (the derivatives of) \( u_1 \) and \( u_2 \):

\[
\begin{align*}
y_1 u_1' + y_2 u_2' &= 0 \\
y_1' u_1' + y_2' u_2' &= f(x)
\end{align*}
\]
Observe that the determinant of the linear system in no other than the Wronskian $W(y_1, y_2) \neq 0$ by assumption. Hence, the system has a unique solution $(u'_1, u'_2)$.

\[
    u'_1 = \frac{\det \begin{pmatrix} 0 & y_2 \\ f(x) & y'_2 \end{pmatrix}}{W(y_1, y_2)} = -\frac{y_2 f(x)}{y_1 y'_2 - y'_1 y_2}
\]

\[
    u'_2 = \frac{f(x) y_1}{y_1 y'_2 - y'_1 y_2}
\]

From $u'_1$ and $u'_2$, we obtain $u_1$ and $u_2$ by integration.
Example

\[ y'' - 3y' + 2y = \frac{e^{3x}}{1 + e^x} \]

Notice that undetermined coefficient methods does not work in this example! Using the characteristic equation technique for instance, we find that the general solution to the associated homogeneous equation is

\[ y_c = C_1 e^x + C_2 e^{2x}. \]
The method of variation of parameters tells us that we can find a particular solution

\[ y_p = u_1 e^x + u_2 e^{2x} \]

by solving

\[
\begin{align*}
e^x u_1' + e^{2x} u_2' &= 0 \\
e^x u_1' + 2e^{2x} u_2' &= \frac{e^{3x}}{1 + e^x}
\end{align*}
\]
\[ u_1' = -\frac{e^{2x}}{1 + e^x} \]

\[ u_2' = \frac{e^x}{1 + e^x} \]

\[ u_1 = -(e^x + 1) + \ln(e^x + 1) \]

\[ u_2 = \ln(e^x + 1) \]

A particular solution is

\[ y_p = (-(e^x + 1) + \ln(e^x + 1))e^x + \ln(e^x + 1)e^{2x} \]

and, finally, the general solution is

\[ y = C_1 e^x + C_2 e^{2x} + e^{2x}(\ln(e^x + 1)) + e^x(\ln(e^x + 1) - (e^x + 1)). \]
Variation of parameters for higher order equations

\[ y^{(n)} + P_1 y^{(n-1)} + \ldots + P_n y = f(x). \]

Let \( y_1, \ldots, y_n \) be \( n \) linearly independent solutions of

\[ y^{(n)} + P_1 y^{(n-1)} + \ldots + P_n y = 0 \]

Look for \( u_1, \ldots, u_n \) such that \( u_1 y_1 + \ldots + u_n y_n \) is a particular solution of the nonhomogeneous equation.
For that, one solves the following linear system:

\[
\begin{align*}
    y_1 u'_1 &+ \ldots + y_n u'_n &= 0 \\
    y'_1 u'_1 &+ \ldots + y'_n u'_n &= 0 \\
    \vdots & \quad \vdots \\
    y^{(n-1)}_1 u'_1 &+ \ldots + y^{(n-1)}_n u'_n &= f(x) \\
\end{align*}
\]

\[
u'_i = \frac{W_i(y_1, \ldots, y_n)}{W(y_1, \ldots, y_n)}
\]

where \( W_i(y_1, \ldots, y_n) \) is the Wronskian in which the column \( i \) has been replaced by the column \((0, \ldots, 0, f(x))\).
Example

\[ y''' + y' = \tan x \]

\[ y''' + y' = \sec x \]
4.5 The Cauchy-Euler equation

Standard form, $n^{th}$ order:

$$a_n x^n \frac{d^n y}{dx^n} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = g(x).$$

Second order example:

$$x^2 y'' - 2xy' + 2y = x^3 \ln x$$

Observation: The substitution $t = \ln x$ reduces Cauchy-Euler equation to an equation with constant coefficients!
From
\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}
\]
and
\[
\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \frac{dt}{dx} = \frac{1}{x^2} \left( -\frac{dy}{dt} + \frac{d^2 y}{dt^2} \right)
\]
Substitution leads to
\[
\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = te^{3t}.
\]
This can be solved using the method of undetermined coefficients or variation of parameters!
\[ y(t) = C_1 e^{2t} + C_2 e^t + \left( \frac{1}{2} t - \frac{3}{4} \right) e^{3t} \]
\[ y(x) = C_1 x^2 + C_2 x + \left( \frac{1}{2} \ln x - \frac{3}{4} \right) x^3 \]
Second order differential equations with constant coefficients

Consider a spring with natural length $L$.

Suspend a mass with weight $F_g = mg$. The spring is stretched by $l$. Choose an orientation vector $\vec{i}$ downward.

Hooke’s Law

$$mg = Kl$$

where $K$ is the constant of the spring.
Disturb the mass $m$ with initial position $x_0$ and velocity $v_0$.

If $x$ denotes the displacement from equilibrium position, then

$$m\ddot{x} = -Kx$$

$$m\ddot{x} = -Kx$$

$$\ddot{x} + \lambda^2 x = 0$$

where

$$\lambda^2 = \frac{K}{m}.$$  

The mass executes a free, undamped motion.

$$x(t) = C_1 \cos \lambda t + C_2 \sin \lambda t.$$
\[ x(t) = A \cos(\lambda t + \Phi) \]

The simple, harmonic motion.

\[ \lambda = \sqrt{\frac{k}{m}} \] is the angular velocity,

\[ T = \frac{2\pi}{\lambda} \] is the natural period of the motion

\[ f = \frac{\lambda}{2\pi} = \frac{1}{T} \] is the natural frequency.

\[ A = \sqrt{C_1^2 + C_2^2} \] is the amplitude and \( \Phi \) from \( \tan \Phi = -\frac{C_2}{C_1} \) is the phase angle.
Solving

\[ \lambda t + \Phi = \frac{\pi}{2} \]

\[ t_0 = \frac{\pi - \Phi}{\lambda} \] is the phase shift.

The motion can be graphed now!
Example #1, page 197.

Example: A 16 lb weight is placed upon the lower end of a coil spring suspended vertically from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. Determine the resulting displacement as a function of time in each of the following cases:

a) If the weight is then pulled down 4 in. below its equilibrium position and released at t=0 with initial velocity of 2 ft/sec directed downward.

b) If the weight is then pulled down 4 in. below its equilibrium position and released at t=0 with an initial velocity of 2 ft/sec directed upward.

c) If the weight is then pushed up 4 in. above its equilibrium position and released at t=0 with an initial velocity of 2 ft/sec. directed downward.
16 = \frac{K}{12} \cdot \frac{6}{12} = \frac{K}{2}, \text{ so } K = 32.

\ddot{x} + \frac{K}{M}x = 0, \quad M = \frac{16}{32} = \frac{1}{2} \text{ slugs}

\ddot{x} + 64x = 0

Finish the example!
Damped motion

\[ M\ddot{x} = -Kx - \beta \dot{x}; \quad F_R = -\beta \dot{x} \]

\[ \ddot{x} + \lambda^2 x + 2b \dot{x} = 0 \]

where

\[ 2b = \frac{\beta}{M}. \]

\[ \ddot{x} + 2b \dot{x} + \lambda^2 x = 0 \]

Free, damped motion The motion is not necessarily periodic anymore.
\[ m^2 + 2bm + \lambda^2 = 0 \]

\[ m = -b \pm \sqrt{b^2 - \lambda^2}. \]

\( b^2 - \lambda^2 > 0 \): Over-damped motion.

\[ x(t) = e^{-bt} [C_1 e^{\sqrt{b^2 - \lambda^2}t} + C_2 e^{-\sqrt{b^2 - \lambda^2}t}]. \]

Observe the damping factor \( e^{-bt} \)!
\[ b^2 - \lambda^2 = 0: \text{Critically damped motion.} \]

\[ x(t) = e^{-bt}(C_1 + C_2 t). \]
$b^2 - \lambda^2 = 0$: Critically damped motion.

$$x(t) = e^{-bt}(C_1 + C_2 t).$$

$b^2 - \lambda^2 < 0$: Under-damped motion.

$$x(t) = e^{-bt}[C_1 \cos \sqrt{\lambda^2 - b^2} t + C_2 \sin \sqrt{\lambda^2 - b^2} t].$$

Observe a periodic factor and a damping factor!
Example #2, page 208.
Forced motion

\[ M\ddot{x} + \beta \dot{x} + Kx = F \cos \omega t \]

\[ \ddot{x} + 2b \dot{x} + \lambda^2 x = f \cos \omega t \]

where

\[ f = \frac{F}{M}, \quad 2b = \frac{\beta}{M}, \quad \lambda^2 = \frac{K}{M}. \]
Assume we are in the under-damped situation, so that

\[ x_c(t) = Ae^{-bt} \cos(\sqrt{\lambda^2 - b^2} t + \phi). \]

We look for a particular solution using the undetermined coefficients method:

\[ x_p(t) = C \cos \omega t + D \sin \omega t \]
We find that:

\[-\omega^2 C + 2b\omega D + \lambda^2 C = f\]
\[-\omega^2 D - 2b\omega C + \lambda^2 D = 0\]
We find that:

\[-\omega^2 C + 2b\omega D + \lambda^2 C = f\]
\[-\omega^2 D - 2b\omega C + \lambda^2 D = 0\]

or equivalently

\[(\lambda^2 - \omega^2)C + 2\lambda\omega D = f\]
\[-2b\omega C + (\lambda^2 - \omega)D = 0\]
By Cramer’s method for instance,

\[ C = \frac{f(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \]

\[ D = \frac{2b\omega f}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \]
\[ x_p = \frac{f(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \cos \omega t + \frac{2b\omega f}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \sin \omega t \]

We will express \( x_p \) as

\[ x_p = A \cos (\omega t + \phi) \]
\[ A \cos \phi = \frac{f(\lambda^2 - \omega^2)}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \]

\[ -A \sin \phi = \frac{2b\omega f}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2} \]

So

\[ A = \frac{f}{[(\lambda^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} \]

and finally

\[ x_p(t) = \frac{f}{[(\lambda^2 - \omega^2)^2 + 4b^2\omega^2]^{1/2}} \cos (\omega t + \phi) \]
\[
\frac{dA}{d\omega} = \frac{2(\lambda^2 - \omega^2)\omega - 4b^2\omega}{[(\lambda^2 - \omega^2)^2 + 4b^2\omega^2]^{3/2}}
\]

The critical \(\omega\)'s are \(\omega = 0\) and \(\omega^2 = \lambda^2 - 2b^2\).

\(x_p(t)\) achieves the maximum amplitude when

\[
\omega = \omega_R = \sqrt{\lambda^2 - 2b^2} = \sqrt{\frac{K}{M} - \frac{\beta^2}{2M^2}}
\]
$\omega_R$ is called the resonance frequency.

Observe that the resonance frequency $\omega_R \leq \sqrt{\lambda^2 - b^2} = \sqrt{\frac{K}{M} - \frac{\beta^2}{4M^2}}$ is smaller than the frequency of the free motion!

The graph of $y = A(\omega)$ is called the resonance curve!
$$x(t) = x_c + x_p$$

The function $x_c(t)$ eventually becomes negligible as time goes on, this is the **transient state**.

$x_p(t)$ which remains forever, is called the **steady state solution**.
In the undamped situation, $b = 0$,

$$x_p(t) = \frac{f}{\lambda^2 - \omega^2} \cos(\omega t + \phi)$$

We can see that as $\omega$ approaches the natural frequency $\lambda$, the amplitude of the steady state blows up! This phenomenon is known as Pure resonance. It always has destructive effects on the systems.
Assignments: page 189, #10-12, Page 217 (Ross), # 4-7, Page 224, #1-3
Example: Forced motion

A mass weighting 4 lb stretches a spring 1.5 in. The mass is displaced 2 in. in the positive direction and released with zero initial velocity. Assuming that there is no damping, and that the mass is acted upon by an external force of $2 \cos 3t$ lb, formulate the initial value problem describing the motion of the mass and solve it.
\[ 4 = \frac{1}{8} K \text{ so } K = 32 \]

\[ x(0) = \frac{1}{6} \]

\[ \dot{x}(0) = 0 \]

\[ 4 = m \cdot 32 \text{ so, } m = \frac{1}{8} \]

\[ \frac{1}{8} \ddot{x} + 32x = 2 \cos 3t \]
\[ \ddot{x} + 256x = 16 \cos 3t \]
\[ x(0) = 0 \]
\[ \dot{x}(0) = 0 \]

Solving the above initial value problem leads to

\[ x(t) = \frac{151}{1482} \cos 16t + \frac{16}{247} \cos 3t. \]
Analogous systems

Mass on a spring:

\[ m\ddot{x} + \beta \dot{x} + Kx = f(t) \]

Inductor-Resistor-Capacitor (L-R-C) series circuit:

\[ L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t) \]

\( E(t) \) is the electric potential)
Put $E(t) = E_0 \sin \omega t$

Write the circuit equation as

$$LI'' + RI' + \frac{1}{C} I = \omega E_0 \cos \omega t$$

Solving:

$$I(t) = I_{tr} + I_{sp}$$

where

$$\lim_{t \to +\infty} I_{tr} = 0$$

$$I_{sp} = \text{Steady periodic solution}$$
\[ I_{sp} = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \]

\[ \alpha = \tan^{-1}\frac{\omega RC}{1 - LC\omega^2}, \quad 0 \leq \alpha \leq \pi \]

\[ Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \]

(in Ohms) is called the **impedance** of the circuit. The amplitude of the signal is

\[ I_0 = \frac{E_0}{Z} \]
Electrical resonance

\[ I_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \]

attains its maximum when

\[ \omega = \frac{1}{\sqrt{LC}} \]

This is the resonance frequency!. Tuning a radio receiver consists essentially in modifying the value of \( C \) so as to match the frequency of the incoming radio signal!
6. Series solutions

Examples

Solve

\[ \frac{dy}{dx} - 2xy = 0 \]

We look for a solution of the form

\[ y = \sum_{n=0}^{\infty} c_n x^n \]

Then

\[ \frac{dy}{dx} = \sum_{n=0}^{\infty} n c_n x^{n-1} \]

and

\[ \frac{dy}{dx} - 2xy = \sum_{n=0}^{\infty} n c_n x^{n-1} - 2 \sum_{n=0}^{\infty} c_n x^{n+1} = 0 \]
\[
\sum_{m=0}^{\infty} (m + 1)c_{m+1}x^m - 2 \sum_{m=1}^{\infty} c_{m-1}x^m = 0
\]

\[
c_1 + \sum_{m=1}^{\infty} [(m + 1)c_{m+1} - 2c_{m-1}]x^m = 0
\]
\[ c_1 = 0 \text{ and } (m + 1)c_{m+1} - 2c_{m-1} = 0 \] which implies that

\[ c_1 = 0 \text{ and } c_{m+1} = 2 \frac{c_{m-1}}{m + 1} \]

for \( m \geq 2 \).

We will use the recurrence formula for \( c_m \) to generate all coefficients in the power series.
\[ \begin{align*}
  c_0 & \quad \text{arbitrary} \\
  c_1 & = 0 \\
  c_2 & = 2 \frac{c_0}{2} = c_0 \\
  c_3 & = 2 \frac{c_1}{3} = 0 \\
  c_4 & = 2 \frac{c_2}{4} = \frac{c_0}{2} \\
  c_5 & = 0
\end{align*} \]
In general,
\[ c_{2n+1} = 0 \]
for \( n = 0, 1, 2 \ldots \) and
\[
\begin{align*}
    c_{2n} &= 2 \frac{c_{2(n-1)}}{2n} \\
    &= 2 \frac{c_{2(n-2)}}{(n-2)(n-1)} = \frac{c_{2(n-2)}}{n(n-1)} \\
    \vdots &= \vdots \\
    &= \frac{c_0}{n!}
\end{align*}
\]
\[ y = c_0 \left( 1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{m!} \right) = c_0 \left( \sum_{m=0}^{\infty} \frac{(x^2)^m}{m!} \right) \]

\[ y = c_0 e^{x^2}. \]
\[ y = c_0 \left( 1 + \sum_{m=1}^{\infty} \frac{x^{2m}}{m!} \right) = c_0 \left( \sum_{m=0}^{\infty} \frac{(x^2)^m}{m!} \right) \]

[Equation]

\[ y = c_0 e^{x^2}. \]

Check this solution using the separable equation technique!
Example 2

\[(x - 1)y'' - xy' + y = 0, \quad y(0) = -2, \quad y'(0) = 6\]

\[y = \sum_{n \geq 0} c_n x^n\]

\[y' = \sum_{n \geq 0} nc_n x^{n-1}\]

\[y'' = \sum_{n \geq 0} n(n - 1)c_n x^{n-2}\]

\[0 = (x - 1)y'' - xy' + y = xy'' - y'' - xy' + y\]

\[= \sum_{n \geq 0} [(n + 1)nc_{n+1} - (n + 2)(n + 1)c_{n+2} + (1 - n)c_n]x^n\]
So

\[ c_{n+2} = \frac{(1 - n)c_n + (n + 1)nc_{n+1}}{(n + 2)(n + 1)} \]

\[ c_0 = y(0) = -2 \]

\[ c_1 = y'(0) = 6 \]

Using the above recurrence formula, we see that

\[ c_2 = -1, \quad c_3 = -\frac{1}{3}, \quad c_4 = -\frac{1}{4.3}, \quad c_5 = -\frac{1}{5.4.3} \]

Claim: for \( n \geq 2 \), \( c_n = -\frac{2}{n!} \).
proof

True for $c_2$ and $c_3$. Assume it is true for $n - 1$ and $n - 2$, then

$$c_n = \frac{(1 - n)c_{n-2} + (n - 1)(n - 2)c_{n-1}}{n(n - 1)}$$

$$= \frac{(1 - n)\frac{-2}{(n-2)!} + (n - 1)(n - 2)\frac{-2}{(n-1)!}}{n(n - 1)}$$

$$= \frac{2}{n!}$$
\[ y = -2 + 6x - 2 \sum_{n \geq 2} \frac{x^n}{n!} \]

\[ = 8x - 2 - 2x - 2 \sum_{n \geq 2} \frac{x^n}{n!} \]

\[ = 8x - 2 \left( \sum_{n \geq 0} \frac{x^n}{n!} \right) \]

\[ y = 8x - 2e^x \]

Check by substitution that this is indeed the solution to the initial value problem.
Example 2

\[(x - 1)y'' - (2 - x)y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1\]

\[c_0 = y(0) = 2, \quad c_1 = y'(0) = -1\]

The recurrence relation is:

\[c_{n+2} = \frac{c_n + (n - 2)c_{n+1}}{n + 2}\]

\[y = 2 - x + 2x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{10} + \frac{x^6}{20} + \frac{x^7}{140} + \ldots\]
Series solutions around ordinary points

\[ y'' + P(x)y' + Q(x)y = 0 \]

The standard form for a second order linear differential equation.

**Definition**

A point \( x_0 \) is said to be an **ordinary point** for the above differential equation if \( P(x) \) and \( Q(x) \) are analytic at \( x_0 \); that is, both have power series in \( (x - x_0) \) with positive radius of convergence.
A point that is not an ordinary point is said to be a singular point.

Note: For power series solutions at an ordinary point, the radius of convergence is at least equal to the distance to the nearest singular point.
Example

\[(x^2 - 1)y'' + 4xy' + \frac{2}{x^2 - 1}y = 0\]

takes the standard form

\[y'' + \frac{4x}{x^2 - 1}y' + \frac{2}{(x^2 - 1)^2}y = 0\]

1 and \(-1\) are singular points for this equation. Any other point is a regular point, according to the above definition.
Theorem

If \( x = x_0 \) is an ordinary point of the differential equation
\[
y'' + P(x)y' + Q(x)y = 0,
\]
we can always find two linearly independent power series solutions of the form

\[
y = \sum_{n=0}^{\infty} c_n (x - x_0)^n
\]
Example

Find two linearly independent solutions for

\[ y'' - xy' + 2y = 0 \]
Example

Find two linearly independent solutions for

\[ y'' - xy' + 2y = 0 \]

Look for

\[ y = \sum_{n=0}^{\infty} c_n x^n \]

Then:

\[ y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \]

\[ y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \]
Therefore, \( y'' - xy' + 2y = 0 \) implies that

\[
0 = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n \\
= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} 2c_n x^n - \sum_{n=1}^{\infty} nc_n x^n \\
= 2c_2 + 2c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} + (2-n)c_n] x^n
\]
We deduce that

\[ c_2 = -c_0 \]

\[ c_{n+2} = \frac{(n - 2)c_n}{(n + 2)(n + 1)} \]
We deduce that

\[ c_2 = -c_0 \]

\[ c_{n+2} = \frac{(n - 2)c_n}{(n + 2)(n + 1)} \]

\( c_0 \): Arbitrary

\[ c_2 = -c_0 \]

\[ c_4 = 0 \]

\[ c_{2n} = 0, \ n > 2 \]
$c_1$ is also arbitrary.

\[ c_{2n+1} = \frac{(2n - 3)c_{2n-1}}{(2n + 1)(2n)}, \quad 2n + 1 > 3 \]

\[ c_3 = -\frac{c_1}{3.2} \]

\[ y_1 = c_0 - c_0 x^2 = c_0 (1 - x^2) \]

\[ y_2 = c_1 x - \frac{c_1}{6} x^3 + \ldots = c_1 (x - \frac{1}{6} x^3 + \ldots) \]
Another example

\[ y'' - xy' - y = 0, \quad x_0 = 1 \]

Find two linearly independent power series solutions.

Look for

\[ y = \sum_{n=0}^{\infty} c_n(x - 1)^n \]

for

\[ y'' - (x - 1)y' - y' - y = 0 \]
6.2. Solutions about singular points: Frobenious Method

Definition

For a differential equation

\[ Py'' + Qy' + Ry = 0 \]

A singular point \( x_0 \) is said to be a regular singular point if \((x - x_0) \frac{Q}{P}\) and \((x - x_0)^2 \frac{R}{P}\) are analytic at \( x_0 \). Otherwise, the regular point \( x_0 \) is said to be irregular singular.
Example: The Euler Equation

\[ x^2 y'' + \alpha xy' + \beta y = 0 \]

\( x = 0 \) is a regular singular point. As an alternate method of solution, we look for

\[ y = x^r \]

Then

\[ x^r (x^r)' \alpha x (x^r)' + \beta x^r = 0 = x^r (r(r - 1) + \alpha r + \beta) \]

Any solution of

\[ F(r) = r(r - 1) + \alpha r + \beta = 0 \]

leads to a solution \( y = x^r \) of the Euler equation.
If $r_1$ and $r_1$ are the 2, real distinct roots, then

$$y = Ax^{r_1} + Bx^{r_2}$$

Example: $2x^2y'' + 3xy' - y = 0$, $x > 0$
Double Roots

\[ F(r) = (r - r_1)^2 \]

In this case \( y_1 = x^{r_1} \) is a solution and the method of reduction of order shows that

\[ y_2 = x^{r_1} \ln x \]

is also a solution. (Check this directly!)

Example:

\[ x^2 y'' + 5xy' + 4y = 0, \quad x > 0 \]
Complex-Conjugate Roots

\[ r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu \]

\[ z_1 = x^{\lambda + i\mu} = e^{(\lambda + i\mu) \ln x} = x^\lambda (\cos (\mu \ln x) + i \sin (\mu \ln x)) \]

\[ z_2 = x^\lambda (\cos (\mu \ln x) - i \sin (\mu \ln x)) \]
\[ y_1 = \frac{1}{2}(z_1 + z_2) = x^\lambda \cos (\mu \ln x) \]

and

\[ y_2 = -\frac{1}{2i}(z_1 - z_2) = x^\lambda \sin (\mu \ln x) \]

are two linearly independent real solutions.

Example:

\[ x^2 y'' + xy' + y = 0 \]
More Generally

Example:

\[ 2x^2 y'' - xy' + (1 + x)y = 0 \]

0 is a regular singular point. We look for

\[ y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \]

Substituting into the differential equation and grouping like terms from lowest power of \( x \) to higher powers:

\[ F(r)x^{r+k} + \sum_{n \geq 1} [G(r, n)] x^{n+r+k} = 0 \]
Solve the indicial equation

\[ F(r) = 0 \]

and obtain recurrence relations from

\[ G(r, n) = 0 \]

If \( r_1 - r_2 \) is not an integer, we always obtain two linearly independent solutions.
Example 2

\[ x^2 y'' + (x^2 + 4x)y' + (2x + 2)y = 0 \]

\[ y = \sum_{n=0}^{\infty} c_n x^{n+r} \]

\[ y' = \sum (n + r)c_n x^{n+r-1}, \quad y'' = \sum (n + r)(n + r - 1)c_n x^{n+r-2} \]
\[ x^2 y' = \sum_{n=0}^{\infty} (n + r) c_n x^{n+r+1} = \sum_{m=1}^{\infty} (m - 1 + r) c_{m-1} x^{m+r} \]

\[ 2xy = \sum_{n=0}^{\infty} 2c_n x^{n+r+1} = \sum_{m=1}^{\infty} 2c_{m-1} x^{m+r} \]
\[ \sum (n + r)(n + r - 1)c_n x^{n+r} + \sum (n + r)c_n x^{n+r+1} + \sum 4(n + r)c_n x^{n+r} + \sum 2c_n x^{n+r+1} + \sum 2c_n x^{n+r} = 0 \]

\[ c_0(r(r - 1) + 4r + 2)x^r + \sum_{m=1}^\infty [[(m + r)(m + r + 3) + 2]c_m + (m + r + 1)c_{m-1}] x^{m+r} = 0 \]

\[ F(r) = r(r + 3) + 2 = 0 = r^2 + 3r + 2 \]

\[ c_m = - \frac{(m + r + 1)c_{m-1}}{(m + r)(m + r + 3) + 2} \]
\[ r_1 = -2, \quad r_2 = -1 \]

Work with the smaller root \(-2\) first:

\[ c_n = -\frac{c_{n-1}}{n} \]

Inductively, we see that

\[ c_n = \frac{(-1)^n c_0}{n!} \]

This leads to

\[ y_1 = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n-2} = c_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = c_0 x^{-2} e^{-x} \]
Form $r = -1$, we obtain

$$c_n = -\frac{c_{n-1}}{n+1}$$

Inductively:

$$c_n = (-1)^n \frac{c_0}{(n+1)!}$$

$$Y_2 = c_0 \sum \frac{(-1)^n}{(n+1)!} x^{n-1} = -c_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} = -c_0 x^{-2} (e^{-x} - 1)$$

Check directly that $x^{-2}$ and $x^{-2} e^{-x}$ are solutions!

The general solution is

$$y = Ax^{-2} e^{-x} + Bx^{-2}$$
6.3 Bessel’s Equation and Bessel’s functions

\[ x^2 y'' + xy' + (x^2 - p^2)y = 0 \]

Bessel’s Equation of order \( p \).

Any solution is called a Bessel function of order \( p \). Theses functions occur in connection with problems of Physics and Engineering.
The case $p = 0$

$$xy'' + y' + xy = 0$$

0 is a regular, singular point.

Look for

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$r^2 c_0 x^{r-1} + (1 + r)c_1 x^r + \sum_{n=2}^{\infty} [(n + r)^2 c_n + c_{n-2}] x^{n+r-1} = 0$$

The indicial equation is

$$r^2 = 0$$

with double root $r_1 = 0 = r_2$. 
Then

\[(1 + r)^2 c_1 = 0\]

and

\[(n + r)^2 c_n + c_{n-2} = 0, \quad n \geq 2\]

\[r = 0 \Rightarrow c_1 = 0\]

\[r = 0 \Rightarrow n^2 c_n + c_{n-2} = 0 \Rightarrow c_n = -\frac{c_{n-2}}{n^2}\]
\[ c_{2n+1} = 0 \]
\[ c_{2n} = \frac{(-1)^n c_0}{(n!)^2 2^{2n}}, \quad n \geq 1 \]

\[ y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{x}{2} \right)^{2n} = y_1(x). \]

\[ J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{x}{2} \right)^{2n} \]

Bessel Function of the first kind of order zero.

\[ J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \ldots \]

A second solution must be of the form (See Theorem 6.3)

\[ y = x \sum_{n=0}^{\infty} c_n^* x^n + J_0(x) \ln x \]
Successive approximations

\[ y' = f(x, y), \quad y(x_0) = y_0. \]

**Picard’s Method**

\[ y = \phi(x), \text{ the d.e. is just specifying the slope of the tangent line to the graph of the solution!} \]

Zeroth approximation: \( \phi_0 = y_0. \)

First approximation: \( \phi_1 \) (satisfying a different d.e.)

\[ \phi_1'(x) = f(x, \phi_0(x)), \quad \phi_1(x_0) = y_0. \]

\[ \phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0)dt \]
Second approximation: $\phi_2$. (Satisfying a different d.e.)

\[
\phi'_2(x) = f(x, \phi_1), \quad \phi_2(x_0) = y_0.
\]

\[
\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\,dt
\]

::

$n^{th}$ approximation:

\[
\phi_n(x) = \int_{x_0}^{x} f(t, \phi_{n-1}(t))\,dt
\]
One has a sequence of functions $\phi_0, \phi_1, \ldots, \phi_n, \ldots$. The exact solution is given by:

$$\phi = \lim_{n \to \infty} \phi_n$$

Picard used this method to prove existence of solutions!

Observe that

$$\phi'(x) = \lim_{n \to \infty} \phi'_n(x) = \lim_{n \to \infty} f(x, \phi_{n-1}(x)) = f(x, \phi).$$
Example

\[ y' = xy, \quad y(0) = 1 \]

Let

\[ y = \phi(x) \]

\[ \phi_0 = 1 \]

\[ \phi_1 = 1 + \int_0^x t \, dt = 1 + \frac{x^2}{2}. \]

\[ \phi_2 = 1 + \int_0^x t(1 + \frac{t^2}{2}) \, dt = 1 + \frac{x^2}{2} + \frac{(\frac{x^2}{2})^2}{2} \]

\[ \phi_3 = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} \]

where \( u = \frac{x^2}{2} \).
\[ \phi_n(x) = \sum_{i=1}^{n} \frac{u^i}{i!} \]

Where \( u = \frac{x^2}{2}. \)

\[ \phi(x) = \lim_{n \to \infty} \phi_n(x) = e^{\frac{x^2}{2}}. \]
Approximating the solution of

\[ y' = f(x, y), \quad y(x_0) = y_0. \]

Let \( x_1 = x_0, x_2 = x_0 + h, \ldots \)

\[ x_N = x_{N-1} + h \]

If \( \phi(x) \) is the exact solution, let \( \phi(x_1), \ldots, \phi(x_N) \) be the evaluation of \( \phi \) at points \( x_k \).
A numerical method will use the IVP to estimate $\phi(x_k)$, $k = 1, 2, ..., N$.

Let $y_1, y_2, ..., y_N$ be approximations to $\phi(x_1), \phi(x_2), ..., \phi(x_N)$. A one-step method uses $y_{k-1}$ to find $y_k$ using the differential equation. The method has also an alternate name of “starting method".
The multi-step methods (using several previous approximations to find $y_k$) are also known as "continuing methods".
The Euler Method

\( \phi \) the exact solution of \( y' = f(x, y) \), \( y(x_0) = y_0 \).

\[ y_{n+1} = y_n + hf(x_n, y_n). \]

Geometrically, the segment of the graph of \( y = \phi(x) \) between \( (x_n, \phi(x_n)) \) and \( (x_{n+1}, \phi(x_{n+1})) \) is replaced by the line segment joining \( (x_n, y_n) \) and \( (x_{n+1}, y_n + hf(x_n, y_n)) \).

\[ y_0 = \phi(x_0) \]
Example

\[ y' = 2x + y; \quad y(0) = 1 \]

Use \( h = 0.2 \).

<table>
<thead>
<tr>
<th>( x_0 = 0 )</th>
<th>( x_1 = 0.2 )</th>
<th>( x_3 = 0.4 )</th>
<th>( x_3 = 0.6 )</th>
<th>( x_4 = 0.8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_0 = 1 )</td>
<td>( y_1 = \frac{12}{10} )</td>
<td>( y_2 = \frac{152}{100} )</td>
<td>( y_3 = \frac{1984}{1000} )</td>
<td>( y_4 = \frac{26168}{10000} )</td>
</tr>
</tbody>
</table>